Numerical Methods

Polynomial :- A function $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots a_n$ $(a_0 \neq 0, n = 0, 1, 2, 3)$

$a_0, a_1 \dots a_n$ are constants. $\boxed{\text{degree} = n}$

eg:- $f(x) = 2$

$f(x) = 2x^0 \longrightarrow$ constant polynomial (degree 0)

$f(x) = 3x + 2 \longrightarrow$ linear polynomial, straight line degree 1

$f(x) = x^2 - 7x + 2 \longrightarrow$ quadratic polynomial, parabola degree 2

$f(x) = x^{3/2} + \frac{1}{\sqrt{x}} + 3 \longrightarrow$ not a polynomial.

If $f(x)$ is a polynomial then $f(x) = 0$ is a polynomial equation.

a) $\longrightarrow$ If $f(x)$ is a polynomial. ~~too~~ Then the equation $f(x) = 0$

b) $\longrightarrow$ If $f(x)$ is a n^{th} degree polynomial.

has n roots.

c) $\longrightarrow$ If $\alpha_1, \alpha_2 \dots \dots \alpha_n$ are the roots of

$a_0x^n + a_1x^{n-1} + \dots a_{n-1}x + a_n = 0$ $a_0, a_1 \dots a_n$ are real constants then.

$$\boxed{\text{Sum of roots} = \sum_{i=1}^{n} (\alpha_i) = \frac{-a_1}{a_0} = -\frac{\text{Coefficient of } (n-1) \text{ power of } x}{\text{Coefficient of } n \text{ power of } x}}$$

$$\boxed{\text{Product of roots} = \prod_{i=1}^{n} (\alpha_i) = (-1)^n \frac{a_n}{a_0} = (-1)^n \frac{\text{Constant term}}{\text{Coefficient of } x^n}}$$

Q) $7x^5 - 2x^3 + 4x^2 - 3x + 7 = 0$

$\sum \alpha_i = 0/7 = 0$

$\prod \alpha_i = (-1)^5 \, 7/7 = \underline{-1}$

Descart's Rule of signs

If $f(x)=0$ is a polynomial equations then

$\rightarrow$ No. of positive real roots of $f(x) \leq$ Numbers of changes of signs in $f(x)$

$\rightarrow$ No. of negative real roots of $f(x) \leq$ Number of changes of sign in $f(-x)$

Example The equation $x^5+x+1=0$ has

a) 4real, 1 complex roots
b) 3real, 2complex roots
c) 2real, 3 complex roots
d) 1real, 4 complex roots.

No real +ve roots

1 or 0 -ve real roots $\rightarrow$ not possible due to complex pair concept. also zero not a root.

$\therefore$ 1 -ve real root

$\therefore$ option d

Algebraic function

$\rightarrow$ A function obtained by applying sum finite no. of algebraic operations on polynomial functions is called an algebraic function.

$\boxed{\text{Addition, subtraction, Multiplication, division, Power}}$ $\rightarrow$ Algebraic operations

Polynomial $\Rightarrow$ Algebraic

Algebraic $\not\Rightarrow$ polynomial.

$\rightarrow$ If $f(x)$ is a algebraic function then $f(x)=0$ an algebraic equation.

Transcendental Equation

$\rightarrow$ A ~~function~~ equation which is not algebraic equation is called transcendental equation

Trignometric, Inverse trignometric, logarithmic, exponential, Hyperbollic, Inverse hyperbollic $\rightarrow$ Transcendtal

$\rightarrow$ Transcendental equations may have no solution or finite no. of solutions or infinite numbers of solutions.

Nature of real roots

→ If $x=\alpha$ is a real root $f(x)=0$, then $f(x)=0$ intercepts the x axis @ $x=\alpha$

→ If $f(x)$ is continous on $[a,b]$ and if $f(a), f(b)$ have opposite signs then there exists atleast one real root lies b/w a & b.

Error of approximation

→ Absolute error $= |\text{Exact error} - \text{approx value}|$

→ Relative error $= \dfrac{|\text{Exact value} - \text{approx value}|}{\text{exact value}}$

→ percentage error $=$ Relative error $\times 100\%$

* If α is an exact value and if x_n is the n^{th} step to find α then error in n^{th} step $\varepsilon_n = |\alpha - x_n| = |x_n - \alpha|$

error in $n+1^{th}$ step $(\varepsilon_{n+1}) = |\alpha - x_{n+1}|$

Order of convergence :-

→ $\boxed{\varepsilon_{n+1} \leq C\, \varepsilon_n^P}$ $\boxed{P \to \text{order of convergence}}$

$C>0$ $P \geq 1$

C is called asymtotic error constant

if P is large then convergence is fast

→ $P=1$ convergence is linear

→ $1<P<2$ then the convergence is super linear

→ $P=2$ convergence is quadratic

→ $P=3$ convergence is cubic

Truncated error

Error obtained by truncating an infinite sum and obtaining a finite sum is called truncation error.

eg. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \infty$

$e^x \simeq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ then remaining terms = error.

ie truncated error $= \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots$

truncated error order = $O(x^4)$ [Big O of x^4]
= order of x^4

Numeric methods of algebraic & Transcendental Equation

If the real root of $f(x)=0$ lies b/w a & b, then we can approximate that real root using the following numerical methods (iterative methods)

- → Bijection method (~~Bolzano's method~~ Bolzana's method)
- ~~Regula (Falsi method)~~
- → Regula-Falsi method (False position method)
- → Newton-Raphson method (Method of Tangents)
- → Secant Method.

BIJECTION METHOD (Bolzana's method)

If the real root of $f(x)=0$ lies b/w a & b

1st approx $x_1 = \dfrac{a+b}{2}$

→ If $f(x_1)=0$ Then x_1 is root

→ If $f(x_1)\neq 0$ then the following cases arise.

if $f(a)$ & $f(x_1)$ have opposite sign then root lies between a, x_1	$x_2 = \dfrac{x_1+a}{2}$
if $f(b)$ & $f(x_1)$ have opposite sign then root lies between b, x_1	$x_2 = \dfrac{x_1+b}{2}$

~~$x_1 = \dfrac{a+b}{2}$~~

~~$x_n = \dfrac{x_{n-1}+a}{2}$ if $f(x_{n-1})$ & $f(a)$ opposite sign~~

~~$x_n = \dfrac{x_{n-1}+b}{2}$ if $f(x_{n-1})$ & $f(b)$ opposite sign~~

~~$x_n = x_{n-1}$ if $f(x_{n-1}) = 0$~~

Q) Find the 2^{nd} approximation of real roots of $xe^{x}-2=0$ b/w 0 & 1 using bisection method.

$a_1=0$ $b_1=1$ $f(x_1)=f(0.5)=0.5e^{0.5}-2<0$		$a_2=0.5$ $b_2=1$ $f(x_2)=f(0.75)=0.75e^{0.75}-2<0$
$a_3=0.75$ $b_3=1$ $f(x_3)=f(0.875)>0$	$a_4=0.75$ $b_4=0.875$ $f(x_4)=f(0.8125)<0$	$a_5=0.8125$ $b_5=0.875$ $f(x_5)=f(0.84375)<0$

$a_6=$

→ Convergence guaranteed

→ very slow

→ Bisection converges with linear order $(P=1)$

→ Length of interval is ~~~~ reduced by half in each iteration

→ Length of interval ab → $\boxed{\frac{b-a}{2^n}}$

→ After n steps → length of interval

→ If ε is a permissible error then $\left|\frac{b-a}{2^n}\right| \leq \varepsilon$

Q Minimum no. of iterations to find real root of $f(x)=0$ in interval $(1,2)$ using bisection method for the error 10^{-4}

$(ab)=(1,2)$

$\varepsilon = 10^{-4}$

$\frac{2-1}{2^n} \leq 10^{-4} \Rightarrow 2^{-n} < 10^{-4} \Rightarrow 2^n \geq 10^4$

~~nlog~~ $n\ln 2 \geq 4\ln 10$

$n \geq \frac{4\ln 10}{\ln 2}$

$n \geq 13.28$

$n = \underline{14}$.

Regular - Falsi method

Q) If the real root of $f(x)=0$ lies between a_1 & b_1

1st approximation

$$x_1 = \frac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)}$$

$$x_n = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}$$

if $f(x_n)$ & $f(a_n)$ opposite signs then $b_{n+1} = x_n$

if $f(x_n)$ & $f(b_n)$ opposite signs then $a_{n+1} = x_n$

if $f(x_n) = 0$ then x_n is root

Q) Find the ~~real~~ first approx of real root of $x^2+x-3=0$ b/w ① & ②

using Falsi method

$f(a_1) = -1$; $f(b_1) = 3$

$a_1 = 1$

$b_1 = 2$

$$x_1 = \frac{1 \times 3 - (2 \times -1)}{3+1} = \frac{3+2}{4} = \frac{5}{4}$$

→ In this method convergence is guaranteed and fast compared to bisection method.

→ This method converges with linear order $P=1$

Newton Raphsons Method / (Method of Tangents)

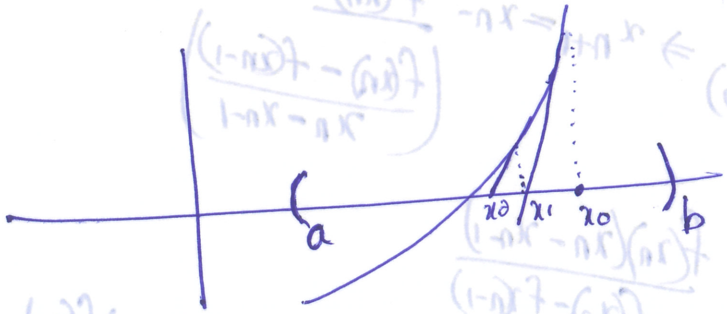
If the real roots of $f(x)=0$ lies b/w a & b

$\rightarrow$ Assume initial approximation x_0 in (a,b)

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$\rightarrow$ 1st approximation $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$\rightarrow$ 2nd approximation $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$



$\rightarrow$ In this method, convergence is guaranted, if we assume, x_0 nearest to the root.

$\rightarrow$ If x_0 is far away from the root then this method may diverge

$\rightarrow$ NR method converges with quadratic order.

$\rightarrow$ ie order of convergences $\underline{P=2}$ quadratic convergence

$\rightarrow$ This method can be used to improve the results of previous methods.

$\rightarrow$ Using this method we can find complex roots also.

$\rightarrow$ if x belongs to (a,b) then NR converges if

$$\boxed{\begin{aligned} &|f(x) \cdot f''(x)| < |f'(x)|^2 \\ \text{ie } &|f(x_0) \cdot f''(x_0)| < |f'(x_0)|^2 \end{aligned}}$$

choose x_0 such that

Secant Method

If $f'(x_n) = 0$ in the neighbourhood then N.R method fails.
Then replace $f'(x_n)$ with the slope of the secant joining the points $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$ in the neighbourhood of x_n.

$$\overbrace{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}^{N.R} \Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{\left(\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}\right)}$$

$$\Rightarrow x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$x_{n+1} = \frac{\cancel{x_n f(x_n)} - x_n f(x_{n-1}) \cancel{-} \cancel{x_n f(x_n)} + (x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})}$$

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$\boxed{x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}}$$

eg:-1 $\quad x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

eg:2 $\quad x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

→ If x_0, x_1 are nearer to the root, then there is a guarantee for convergence. Otherwise this method may diverge.

→ Secant method is a modified version of Regula Falsi method.
ie) To apply secant method formula. Find first two approximations using regula falsi method.

→ Order of converges   super linear order

$$\boxed{P = 1.62}$$

$$\boxed{\begin{matrix} P>1 \\ P<2 \end{matrix}\Big\} 1<P<2 \quad \text{order of convergence } P}$$

Q) Find the second approximation of root of $x^2 - 2 = 0$ by taking $x_0 = -1$ (Using NR method)

$$f(x) = x^2 - 2$$

$$f'(x) = 2x$$

$$x_1 = -1 \frac{+1}{-2} = \frac{-3}{2} = \underline{\underline{-1.5}}$$

$$x_2 = -1.5 - \frac{2.25-2}{-3}$$

$$x_2 = -1.5 + \frac{0.25}{3} = \frac{-4.25}{3} = \underline{\underline{-1.41666}}.$$

Q) Find N-R iterative formula for $\sqrt[3]{5}$

Solution Let $x = \sqrt[3]{5}$

$$x^3 = 5$$

$$\Rightarrow x^3 - 5 = 0$$

$$f(x) = x^3 - 5 = 0$$

$$f'(x) = 3x^2$$

By N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \left(\frac{x_n^3 - 5}{3x_n^2}\right)$$

$$x_{n+1} = \frac{3x_n^3 - x_n^3 + 5}{3x_n^2} = \frac{2x_n^3 + 5}{3x_n^2}$$

* Note taking $f(x) = x - 5^{1/3}$ does not work because $f'(x) = \text{constant} \Rightarrow$ No iteration.

Q) The N-R iterative formula $x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$ can be used to solve a) $x^3 - R = 0$ b) $x^3 + R = 0$ c) $x^2 - R = 0$ d) $x^2 + R = 0$

Solution $\boxed{\text{as } n \to \infty \quad x_{n+1} = x_n = x}$ Concept!

$$\Rightarrow x = \frac{1}{2}\left(x + \frac{R}{x}\right)$$

$$= 2x^2 = x^2 + R$$

$$x^2 = R$$

$$\Rightarrow \underline{x^2 - R = 0} \quad \text{option } \underline{c}.$$

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Q1) Using bisection method, find the 2nd approximation of root of $x^3+x^2+x+7=0$ b/w -3 & -2

<u>Solution</u>

$x_1 = \underline{-2.5}$

$f(x_1) = -4.875$

$f(-3) = -27+9-3+7$

$= -14$

$f(-2) = -8+4-2+7$

$f(-2) = \underline{1}$

$x_2 = \dfrac{-2.5-2}{2} = -2.25$

$f(x_2) = f(-2.25) =$

Q2) Using False position method find 2nd appr. of $x^3+x-1=0$ b/wt 0.5 & 1

$f(1) = 1 \underline{>0}$

$f(0.5) = 0.125+0.5-1$

$= -0.375 \underline{<0}$

$x_1 = \dfrac{(1\times -0.375) - 0.5 \cdot 1}{-0.375-1}$

$x_1 = \dfrac{~~-0.875~~}{~~-1.375~~}$

$x_1 = \dfrac{0.875}{1.375} = 0.636$

$f(0.636) = \underline{-0.106} < 0$

$x_2 = \dfrac{1(-0.106) - (0.636) 1}{-0.106-1} = \underline{0.67}$

③ If we apply secant method to find the root of $xe^x = 2$ b/w 0 and 1. The value after 2 steps is

<u>Solution</u> Initial values not given to apply secant method iterative formula.

So in the interval 0, 1 first two values

④ If $x_0 = -2$ is an initial approximation for the real root of $x^5 - 10x + 100 = 0$. Then the first approximation of root of f(x) by N-R method

Sol) $f(x) = x^5 - 10x + 100$

$f'(x) = 5x^4 - 10$

$x_0 = -2$

$x_1 = x_0 - \frac{f(-2)}{f'(-2)} = -2 - \left(\frac{-32+20+100}{70}\right)$

$x_1 = -2 - \frac{88}{70} = \underline{-3.2057}$

5) Find first approximation of root of $x^3 - 5x^2 + 6x - 1 = 0$ with initial guess $x = 1$ using N.R method.

Sol) $f(x) = x^3 - 5x^2 + 6x - 1$

$f'(x) = 3x^2 - 10x + 6$

$x_0 = 1$

$x_1 = +1 - \left(\frac{1-5+6-1}{3-10+6}\right)$

$x_1 = +1 - \frac{1}{-1}$

$x_1 = +1+1 = \underline{2}$

6) & 7) Find N-R iterative formula for $\sqrt[5]{N}$ $(N>0)$ ⁶) for $N = 30$ and $x_0 = 2$ ⁷)

Find first approximation $\sqrt[5]{30}$ by NR formula.

$x = \sqrt[5]{N}$

$x^5 = N$

$f(x) = x^5 - N = 0$

$f'(x) = 5x^4$

$x_{n+1} = x_n - \frac{(x_n^5 - N)}{5x_n^4}$

$x_{n+1} = (5x_n^5 - x_n^5 + N) / 5x_n^4$

$$\boxed{x_{n+1} = \frac{4x_n^5 + N}{5x_n^4}}$$

⇓

7) $x_1 = \frac{4x_0^5 + 30}{5(x_0)^4} = \frac{4 \times 32 + 30}{5 \times 16}$ $\quad x_0 = 2$

$x_1 = \frac{128+30}{80} = \frac{158}{80} = \underline{1.975}$

8) The N-R formula $x_{n+1} = \frac{2x_n^3+1}{3x_n^2+1}$ can be used to solve

Sol) $x = \frac{2x^3+1}{3x^2+1}$

$3x^3 + x - 2x^3 - 1 = 0$

$\underline{x^3 + x - 1 = 0}$

option a.

Numerical Solutions of O.D.E

→ If $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$ is an initial value problem then we can find $y(x_1)$, $y(x_2)$ and so on ... $y(x_n)$ using the following Numerical Methods.

1) Taylor Series Method
2) Runge Kutta Method

$\Big\}$ Single step Methods

3) predictor - corrector Method → Multistep.

TAYLOR SERIES METHOD

If $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$ then

$$h = x_1 - x_0$$

$$y(x_1) = y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots\dots$$

$$h = x_2 - x_1$$

$$y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1'' + \dots\dots$$

Ex If $\frac{dy}{dx} + 2xy = 4$ with $y(0) = 0.2$ Find $y(0.2)$ using Taylor series up to h^3 terms.

taylor series using.

$x_0 = 0$ $\quad x_1 = 0.2$

$y_0 = 0.2$

$h = 0.2 - 0 = 0.2$

~~$y(0.2) = 0.2 + \frac{0.2}{1!}$~~

$y' = 4 - 2xy$

$y'' = 0 - 2xy' - 2y$

$y''' = -2[xy'' + y' + y']$

$y'(0) = 4 - 2(0 \times 0.2) = 4$

$y''(0) = -2[0 + 0.2] = -0.4$

$y'''(0) = -2[0 + 2 \times 4] = \underline{\underline{-16}}$

$$y(0.2) = 0.2 + \left(\frac{0.2}{1!}\right) \times 4 + \frac{0.2^2}{2!} \times -0.4 + \frac{(0.2)^3}{3!} \times -16$$

$$y(0.2) = 0.2 + 0.8 + (-0.008) + -0.21333 = \underline{\underline{0.970}}$$

→ Taylor series method is a single step method.
→ This method gives highest accuracy for more no. of terms.
→ This method is not suitable for computer based solutions.

Range-kutta methods (R-K method)

Forward Euler's method (1st order R-K method)

If $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$

$$y_1 = y(x_1) = y_0 + h\,f(x_0, y_0) \qquad \frac{dy}{dx} = f(x,y)$$

$$y_2 = y(x_2) = y_1 + h\,f(x_1, y_1) \qquad y_0' = f(x_0, y_0)$$

$$y_1 = y(x_1) = y_0 + h y_0'$$

$$y_2 = y(x_2) = y_1 + h y_1'$$

→ The value of y in euler's method is equal to $(h)^1$ terms in taylor series (ie first two terms)

ie) By Taylor series

$$\underbrace{\boxed{y_1 = y(x_1) = y_0 + \frac{h^1}{1!} y_0'}}_{\text{Euler's method}} + \underbrace{\frac{h^2}{2!} + y''_0 + \frac{h^3}{3!} y'''(0)}_{\text{Truncated error}} + \cdots\cdots$$

$\therefore$ Truncated error in Euler method $= \frac{h^2}{2!} + y''(0) + \frac{h^3}{3!} y'''(0) + \frac{h^4}{4!} y^{iv}(0) + \cdots$

$= O(h^2)$

$=$ order of h^2

Stability Condition in Euler's Method

If $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$. Then Euler's method gives stable solution if $\boxed{\left|1 + h\frac{\partial f}{\partial y}\right| < 1}$

Example if $\frac{dy}{dx} = \frac{y-x}{x+y}$ with $y(0)=1$ find $y(0.1)$ by taking $h=0.1$ using Euler's method.

h → step size.

Solution. $x_0 = 0$, $y_0 = 1$

$y' = \frac{y-x}{x+y}$

$y_0' = f(x_0, y_0) = \frac{1-0}{1+0} = 1$

$y(1) = y(0.1) = 1 + (0.1 \times 1) = \underline{1.1}$

Q) If $\frac{dx}{dt} = -3x+2$ with $x(0)=1$ the solution can be solved by Euler's method. The largest time step required without making unstable solution is,

if $\left|1 + h\frac{\partial f}{\partial x}\right| < 1$

Solution.

$f(x,t) = -3x+2$

$\frac{\partial f}{\partial x} = \underline{\underline{-3}}$

$|1 + h(-3)| < 1$

$1 + h - 3 = -1$

$-3h = -2$

$h = -2/-3 = \underline{\underline{0.66}}$

MODIFIED EULER's METHOD or ~~Heun's method~~

Second order R-K method. **Heun's method**

If $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$

$$\boxed{y_1 = y(x_1) = y_0 + \frac{1}{2}(k_1 + k_2)}$$ where $$\boxed{\begin{aligned} k_1 &= h f(x_0, y_0) \\ k_2 &= h f(x_0 + h, y_0 + k_1) \end{aligned}}$$

3rd order R-K method

$$y' = \frac{dy}{dx} = f(x,y) \quad \text{with } y(x_0) = y_0$$

$$\boxed{y_1 = y(x_1) = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)}$$

$$\boxed{\begin{aligned} k_1 &= hf(x_0, y_0) \\ k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\ k_3 &= hf(x_0 + h, y_0 + k_2) \end{aligned}}$$

4th order R-K method

$$y' = \frac{dy}{dx} = f(x,y) \; ; \; y(x_0) = y_0$$

$$\boxed{y_1 = y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$\boxed{\begin{aligned} k_1 &= hf(x_0, y_0) \\ k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\ k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\ k_4 &= hf(x_0 + h, y_0 + k_3) \end{aligned}}$$

- All R-K Methods are single step methods.
- R-K Methods are suitable for computer based solutions.
- R-K methods are self starting methods.

MULTISTEP METHODS

→ If y_0, y_1, y_2, y_3 are initial values we can find y_n using the following predictor corrector method.

→ Adams-Bashforth-Moulton predictor corrector method

→ Milne-Simpsons predictor corrector method

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(18) Solve by Taylor Series upto h^2 term if $\frac{dy}{dx} + 2xy = 4$ with $y(0) = 0.2$ Find $y(0.2)$

(19) $\frac{dy}{dx} + 2xy = 4$ with $y(0) = 0.2$, $y(0.2) = ?$ by Euler

(22) given $\frac{dy}{dx} = x+y$, $y(0) = 0$. Compute $y(0.2)$ using modified Euler's method with $h = 0.2$

given $y(0)=0 \Rightarrow x_0 = 0 \quad y_0 = 0$

$y(0.2) = x_1 = 0.2 \quad y_1 = ?$

$k_1 = h(f(0,0)$
$= 0.2(0,0)$
$= 0$

$k_2 = h[f(0+0.2), 0+0)$
$= 0.2(0.2)$
$= 0.04$

$y_1 = y(0.2) = y_0 + \frac{k_1 + k_2}{2}$
$= 0 + \frac{0.04}{2}$

$y_1 = \underline{0.02}$.

(Q24) $\frac{dy}{dx} + 2xy = 4$; $y(0) = 0.2$ using R-K method find the value of $y(0.2)$ with $h = 0.2$

if nothing mentioned use advanced i.e 4^{th} order

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
$= 0.2 + \frac{1}{6}(0.8 + 4 \times 0.776 + 0.72192)$

$y_1 = \underline{0.970}$

$k_1 = hf(x_0, y_0)$
$k_1 = 0.2 \times 4$
$k_1 = 0.8$
$k_2 = h\, f(0.1, 0.6)$
$k_2 = 0.2(4 - 0.12)$
$k_2 = 0.2(3.88)$
$k_2 = \underline{0.776}$

$f(x,y) = 4 - 2xy$

$k_4 = h\, f(0.2, 0.776)$
$k_4 = 0.2(4 - (2 \times 0.2 \times 0.776))$
$k_4 = \underline{0.72192}$

$k_3 = 0.2\, f(0.1, 0.588)$
$k_3 = 0.2(4 - 2 \times 0.1 \times 0.588) = 0.776$

To find iterative formula for reciprocal of a number

put $\frac{1}{x} = N \quad \frac{1}{x} - N = f(x)$

$f'(x) = -\frac{1}{x^2}$

Only Bisection cannot be used for complex roots.

To find inverse square root of N

$x = \frac{1}{\sqrt{N}} \quad \frac{1}{x^2} = N$

$\frac{1}{x^2} = f(x)$

$f'(x) = \frac{-2}{x^3}$

LINEAR ALGEBRA

① Introduction
② Determinant
③ Rank of Matrix
④ System of Equations
⑤ Eigen values & Eigen Vectors
⑥ Cayley Hamilton Theorm.

Matrix

$(m \times n)$ (row x column) a_{ij} i^{th} row & j^{th} column element.

→ A arrangement real or complex nos in rows or columns is known as matrix.

$m = n =$ square matrix

$m \neq n =$ Rectangular matrix — $m > n$ vertical matrix; $m < n$ Horizontal matrix

$m = 1$ — row matrix

$n = 1$ — colum matrix

→ $(a_{ij} = 0 \;\; i > j)$ Upper triangular

→ $(a_{ij} = 0 \;\; i < j)$ Lower triangular

} both → diagonal matrix $\begin{cases} a_{ij} = k_i \; \forall i=j \\ a_{ij} = 0 \\ j \neq i \end{cases}$

→ $a_{ij} = k$ $i = j$; $a_{ij} = 0$ $i \neq j$ } Scalar matrix, if $k = 1$ ⇒ unit Matrix

→ $a_{ij} = 0$ } Null matrix.

→ Idempotent Matrix $(A^2 = A)$ if square.

→ Nilpotent Matrix $(A^m = 0)$ $m =$ index of matrix } square. $\boxed{m \leq \text{order}}$

both have $|A| = 0$, $A = 0$

eigen Value $= 0$.

→ Involoutary Matrix $A^2 = I$

→ Periodic $A^{m+1} = A$

$m \to$ period of m.

$\boxed{\text{period } 1 \Rightarrow \text{Idempotent}}$

Symmetric Matrix

if $A^T = A$ ie) $A_{ij} = A_{ji} \ \forall \ i,j <$ order square!

Skew Symmetric Matrix

$A^T = -A \Rightarrow a_{ij} = -a_{ji} \ \forall \ i,j <$ order square!

diagonal elements compulsary zero.

Orthogonal Matrix

$A \cdot A^T = I$ or $A^T = A^{-1}$

eg:- $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Properties

→ The diagonal elements of skew ~~sym~~ symmetric matrix is zero.

→ The sum of elements of skew symmetric matrix is zero.

→ If a square matrix $A_{n\times n}$ is defined as $a_{ij} = i^m - j^m \ \forall \ i,j$ then A is skew symmetric.

$a_{ij} = i^m - j^m$

$a_{ij} = -(j^m - i^m)$

$a_{ij} = -(a_{ji})$

→ The determinant of odd order skew symmetric is zero

$A^T = -A$

$|A^T| = -|A|$

$|A| = -|A|$

$\Rightarrow |A| = 0$

→ Determinant of ~~equal~~ even order skew symmetric is a perfect square.

→ Determinant of an orthogonal matrix is ± 1

$A^T = A^{-1}$

$|A| = \frac{1}{|A|}$

$|A|^2 = 1 \Rightarrow |A| = \pm 1$

→ The no of independent elements in a symmetric matrix of $A_{n\times n}$ is $\frac{n(n+1)}{2}$

→ No. of independent elements in a skew symmetric matrix.

$\frac{n(n+1)}{2} - n = \frac{n^2 + n - 2n}{2} = \frac{n(n-1)}{2}$

Scalar — 1 } independent
diagonal — n } element

→ Every Square matrix can be expressed as the sum of symmetric & skew skew symmetric matrix uniquely.

$$A = \left(\frac{A+A^T}{2}\right) + \left(\frac{A-A^T}{2}\right) \text{ } \} \text{ uniquely.}$$

$\downarrow$ symmetric, $\downarrow$ skew-symmetric.

A = Symmetric

Q) if $A_{5\times5} = [a_{ij}]$ such that $a_{ij} = i^2 - j^2 \ \forall i, j$

then $\left[\sum_{i,j=1}^{5} (a_{ij} = ?)\right]$ zero

Q Same quest $|A| = ?$ zero.

Q) if $A = \begin{bmatrix} 1 & 8 & -3 \\ 2 & 5 & 6 \\ 5 & -4 & 6 \end{bmatrix}$ then corresponding skew symmetric matrix is.

$$\left(\frac{A-A^T}{2}\right) = \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & 5 \\ 4 & -5 & 0 \end{bmatrix}$$

Q) How many no. of symmetric matrices can be formed with $(0, \pm1, \pm2)$

n-order $\frac{n(n+1)}{2}$ – independent elements.

5 - choices ∴ $5^{\left(\frac{n(n+1)}{2}\right)}$

→ each inpendent elements can take 5 possible values ∴ $\frac{n(n+1)}{2}$ elements = $5^{\frac{n(n+1)}{2}}$

→ n element can take 5^n possible combination.

⇓

Skew symmetric $5^{\frac{n(n-1)}{2}}$

diagonal 5^n

Scalar — 5

Q) How many symmetric. if {0, 1, 2}

$= 3^{\frac{n(n+1)}{2}}$

How many skew symmetric.

= 1 {all element zero}

$\{0, \pm 1, 2\}$

Square $= 4^{\frac{n(n+1)}{2}}$

Skew $= 3^{\frac{n(n-1)}{2}}$

$\{\pm 1, \pm 2\}$

Square $= 4^{\frac{n(n+1)}{2}}$

Skew $= 0$

Q) $A_{3\times 2}$ $B_{2\times 4}$ AB how many multiplications ——?

Each element in result $= (AB)_{3\times 4}$ $\therefore$ $\underline{12 \times 2 = 24}$

is got by 2 multiplication and one addition.

$\boxed{A_{m\times n} \text{ \& } P_{n\times p} \text{ then } (A\cdot P)_{m\times p} \Rightarrow \begin{array}{l} mnp \text{ multiplication} \\ m(n-1)p \text{ additive} \end{array}}$

Minor of an element. $[M_{ij}]$ standard notation (only for square)

→ The determinant of the remaining matrix after removing i^{th} row & j^{th} column is called minor (M_{ij}) of the element a_{ij} of a square matrix.

Cofactor of a_{ij} :-

$$\text{Cofactor}(a_{ij}) = (-1)^{i+j} \times M_{ij}$$

Cofactor Matrix.

Every element replaced with it's cofactor.

$$\begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

Adjoint of Matrix Adj (A)

$$\text{Adj } A = (\text{Cofactor of } A)^T$$

$$\begin{bmatrix} M_{11} & -M_{21} & M_{31} \\ -M_{12} & M_{22} & -M_{32} \\ M_{13} & -M_{23} & M_{33} \end{bmatrix}$$

Determinant of $(n \times n)$ matrix

function : $(n \times n \longrightarrow \mathbb{R})$

eg:- $|A| = a_{11}M_{11} + a_{12}(-M_{12}) + a_{13}(M_{13})$

Sum of elements of a $\boxed{\text{row}}$ multiplied with corresponding cofactors

OR Sum of elements of a $\boxed{\text{column}}$ multiplied by corresponding cofactors.

> 2n way to expand determinant.
> $n!$ terms in a determinant.
> each term has n multiplication factors
> $\therefore n \cdot n!$ total multiplication required.
> $n!$ ~~multiplication~~ addition required.
>
> $A_{n \times n}$

Inverse of a matrix

if $AB = BA = I$ then A is inverse of B

$\Rightarrow A^{-1} = B$ then B is inverse of A

$\Rightarrow B^{-1} = A$ then A & B are called inverse matrices to each other.

Property

$\boxed{\text{Multiplication associative}}$

$((AB)\cdot(CD))^{-1}$
$= (CD)^{-1} \cdot (AB)^{-1}$
$= D^{-1} C^{-1} B^{-1} A^{-1}$

- $(AB)^{-1} = B^{-1} A^{-1}$
- $(ABCD)^{-1} = \cancel{(DCBA)^{-1}}\ D^{-1} C^{-1} B^{-1} A^{-1}$
- $A \dfrac{(\text{Adj} A)}{|A|} = I = \dfrac{\text{Adj}(A)}{|A|} A$

$\Rightarrow \boxed{A^{-1} = \dfrac{\text{Adj}(A)}{|A|}} \Rightarrow \boxed{[\text{Adj} A]^{-1} = \dfrac{A}{|A|}}$

if $|A| = 0 \Rightarrow A$ is singular matrix A^{-1} doesnt exist.
A is not invertable

- If $|A| \neq 0 \Rightarrow A$ is non singular matrix

Q) $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & -1 & 5 \\ 1 & 2 & 4 \end{bmatrix}$ $\boxed{A^{-1} = ?}$

$|A| = 1(-4-10) + 3(-5) + 2(+1)$

$|A| = -14 - 15 + 2 = -27$

Cofactor $\begin{bmatrix} -14 & 5 & 1 \\ 16 & 2 & -5 \\ 13 & -5 & -1 \end{bmatrix}$ Adj $= \begin{bmatrix} -14 & 16 & 13 \\ 5 & 2 & -5 \\ 1 & -5 & -1 \end{bmatrix}$

Q) $A = \begin{pmatrix} 1 & -1 & 3 \\ -2 & 0 & 4 \\ 2 & -2 & 6 \end{pmatrix}$

Adj A =

$|A| = 0$

Since $R_3 = 2R_1$

$CoA = \begin{vmatrix} +8 & 4 & 4 \\ 0 & 0 & 0 \end{vmatrix}$

Properties

- Determinant of diagonal, triangular, & scalar is product of diagonal elements.
- If any two rows or columns are identical then its determinant is zero.
- If any two rows or columns are proportional then determinant is zero.
- If any two rows/columns are interchanges then determinant is absolutely same with different sign.
- Adding a row to ~~one~~ another row ~~after~~ after multiplied by a non zero scalar. then determinant remains same.

$R_j \longrightarrow R_j + kR_i \Longrightarrow |A| = \text{same}.$

if any row/column is multiplied by k then determinant becomes k times

- $|I_{n\times n}| = 1$
- $|AB| = |A|\cdot|B|$
- $|kA| = k^n |A| \quad (A_{n\times n})$

- $|A^n| = |A|^n$
- $|A| = |A^T|$
- $|A^{-1}| = \frac{1}{|A|}$

- $|\text{adjoint of } (A)| = |A|^{n-1}$

$$\boxed{\begin{aligned} \text{adj}(A) &= |A|\, A^{-1} \\ \Rightarrow |\text{adj}(A)| &= |A|^n\, |A^{-1}| \\ \Rightarrow |\text{adj}(A)| &= |A|^n \cdot \frac{1}{|A|} \\ \Rightarrow |\text{adj}(A)| &= |A|^{n-1} \end{aligned}}$$

$$(\text{Adj } A)^{-1} = \frac{\text{Adj}(\text{Adj } A)}{|\text{Adj } A|}$$

- $|\text{Adj}(A^{-1})| = |(\text{Adj}(A))^{-1}| = |A|^{1-n}$
- $|\text{adj}(kA)| = k^{n(n-1)} |A|^{n-1}$
- $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

Q) $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = ?$

$C_1 \to C_1 + C_2 + C_3 + C_4$

$$\begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix}$$

$R_2 - R_1$; $R_3 - R_1$; $R_4 - R_1$ $\Rightarrow$

$$\begin{vmatrix} x+3a & a & a & a \\ 0 & (x-a) & 0 & 0 \\ 0 & 0 & (x-a) & 0 \\ 0 & 0 & 0 & (x-a) \end{vmatrix}$$

$|A| = (x+3a)(x-a)^3$

try to reduce any one row or column to a single element row/column with rest all zeroes.

Q) $A_{3\times3} = (a_{ij})_{3\times3}$ such that $|A| = 5$

then $|2A| = ?$

$|A| = 5 \quad |2A| = 2^3|A| = 8\times5 = \underline{40}$

Q $A_{3\times3} = (a_{ij})_{3\times3}$ & $B = (b_{ij})_{3\times3}$ such that $b_{ij} = 2^{i+j}(a_{ij})$

and $|A| = 2$ then $|B| = ?$

$$|B| = \begin{vmatrix} 4a_{11} & 8a_{12} & 16a_{13} \\ 8a_{21} & 16a_{22} & 32a_{23} \\ 16a_{31} & 32a_{32} & 64a_{33} \end{vmatrix} = \begin{matrix} 2^2 \\ 2^1 \end{matrix} \begin{vmatrix} 16 & 18 & 64 \\ 16 & 32 & 64 \\ 16 & 32 & 64 \end{vmatrix}$$

$= 2^3 \cdot 2^4 \cdot 2^5 \cdot 2^6 = 2^{12}|A|$

$= \underline{\underline{2^{13}}}$

choose $A = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$B = \begin{vmatrix} 8 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 64 \end{vmatrix} = 2^3\times2^4\times2^6 = \underline{\underline{2^{13}}}$

Q) $A_{10\times5}$, $B_{5\times20}$, $C_{20\times10}$ how many min no. of multiplication needed.

to compute ABC

$(AB)C = 10\times5\times20 + 10\times20\times10$

$(AB)C = 3000$

$A(BC) = \cancel{}\ 5\times20\times10 + 10\times5\times10$

$A(BC) = 1000 + 500 = \underline{\underline{1500}}$

- A matrix is obtained from $A(m\times n)$ by leaving some rows or columns or both. **SUBMATRIX**

No. of possible sub matrix = ~~$1 + (m+n) + (m+n)(m+n-1)\dots$~~

$$\boxed{1 + {}^{(m+n)}C_1 + {}^{m+n}C_2 + \dots + {}^{m+n}C_{m+n} = 2^{m+n}} \quad ★★★$$

Every matrix is sub matrix of itself.

Minor of matrix
The determinant of a square sub matrix is called minor of the matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix} \quad 3\times3 \text{ minor.}$$

$$\begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \quad 2\times2 \text{ minor.}$$

$$|a_{12}| \quad 1\times1 \text{ minor.}$$

Note:- Minor of a matrix is not unique but minor of an element is unique. (only sq. matrix)

no. of 3×3 minors in a matrix of A_{pq} = ${}^pC_3 \times {}^qC_3$

$$\boxed{\text{In general no. of } r\times r \text{ minors in } [A_{m\times n}] \text{ matrix} = {}^mC_r \cdot {}^nC_r}$$

→ for $r\times r$ minor r rows must be selected from m total rows — ${}^mC_r$ ways.
r columns must be selected from n total columns — ${}^nC_r$

eg:-
$(1\times1) = 3C_1 \times 4C_1 = m\cdot n = 12$
$(2\times2) = 3C_2 \times 4C_2 = 3\times6 = 18$
$(3\times3) = 3C_3 \times 4C_3 = 1\times4 = 4$
$\quad\quad 34$

Rank of Matrix $\rho(A)$ or $r(A)$

all minor $= 0$ then Vanishing.

- An order of highest non vanishing minor (non zero) of a matrix $A_{m \times n}$ is called rank of matrix
- An non negative integer r is said to be rank of matrix if atleast one $r \times r$ minor is non zero. (given $r+1$ order minor doesnot exist) if it exist then it should be zero.
- For square matrix if determinant ~~Rank~~ is non zero then rank = order $\boxed{(n)}$ (because order $= n$, $n \times n$ minor is vanishing otherwise)

eg:- $\begin{bmatrix} 3 & 4 & -5 \\ 0 & 1 & 3 \\ -1 & 2 & 4 \end{bmatrix}$

$3 \times (4-6) - 4(3) - 5(+1)$

$= -6 - 12 - 5 = -23 \neq 0$

rank $= 3$.

eg. $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 3 \\ 5 & 8 & 5 \end{bmatrix}$

$C_3 = -C_1$

$|A| = 0$

rank < 3

$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$

$\therefore$ rank $= 2$.

eg. $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ rank $= 2$

Elimentary transformations

1. Interchanging of any two rows. $R_i \leftrightarrow R_j$, $C_i \leftrightarrow C_j$ $\Rightarrow$ sign changes of det.
2. Multiplication of a row or column with a non zero scalar. $R_i \longrightarrow kR_i$ Det $\Rightarrow$ K times
3. Adding of a row to another row after multiplication by a non zero scalar $R_j \longrightarrow R_j + kR_i$ (det not changing

Equalent Matrices (~)

→ Two matrices A and B are equalent to each other if one is obtained from other, by applying a sequence of elementary transformations.

→ $\boxed{\text{Rank of equalent matrices are same.}}$

Elimentary Matrix

A matrix is obtained from unit matrix by applying (any) a single elementary transformation is called elimentary matrix.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

elimentary matrix.

Echelon form

A matrix is said to be in echelon form if it satisfies

→ ① If zero row exist then it should be below of all non zero rows

→ ② The no. of zeroes before first non zero element in every row ~~in every row~~ is less than such no of zeroes in the next row.

$$eg:- \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Always diagonal matrix & upper triangular matrix is in echlon form.

Note: Echelon form of a square matrix is an upper triangular matrix

- Every matrix can be reduced to echelon form by applying elementary row transformations.

- If an matrix is in echelon form then rank of A matrix is equal to the no. of non zero rows. [is equal to no. of independent rows/independent columns]

$\boxed{\text{No. of independent row/column} = \text{Rank of A}}$

Properties

→ Rank of matrix is non negative integer (whole number)

→ $\rho(0 \text{ matrix}) = 0$

→ rank of unit matrix $= I$ $\quad \rho(I) = n \quad \rho(I_{n\times n}) = n$

$\boxed{\rho(A) = \rho(A^T)}$

→ $\rho(A_{m\times n}) \leq \min(m, n)$

→ $\rho(A_{n\times n}) = n$ if $|A_{n\times n}| \neq 0$ Non singular

→ $\rho(A_{n\times n}) < n$ if $|A_{n\times n}| = 0$ Singular.

→ $\rho(A+B) \leq \rho(A) + \rho(B)$

→ $\rho(A-B) \geq \rho(A) - \rho(B)$

→ $\rho(AB) \leq \text{Min of } (\rho(A), \rho(B))$

→ $\rho(A_{n\times n}) = n \Rightarrow \rho(\text{Adj } A) = n$

→ $\rho(A_{n\times n}) = n-1 \Rightarrow \rho(\text{Adj } A) = 1$

→ $\rho(A_{n\times n}) \leq n-1 \Rightarrow \rho(\text{Adj } A) = 0$

eg:- $\rho(A_{6\times 6}) = 4 \Rightarrow \rho(\text{Adj } A) = 0$

$\rho(A_{5\times 5}) = 4 \Rightarrow \rho(\text{Adj } A) = 1$

Q) $AX \begin{bmatrix} 3 & 4 & -5 \\ 2 & -1 & 3 \\ 7 & 6 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -5 \\ 2 & -1 & 3 \\ 4 & 2 & 0 \end{bmatrix}$ then $A = ?$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$RHS = R_3 \to R_3 - R_1$ ∴

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_3 \to R_3 - R_1$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

elimentary matrix obtained by applying same transformation

$\boxed{\text{every elimentary row transformation is nothing but matrix multiplied with corresponding elementary matrix}}$

Q) $\begin{bmatrix} 1 & 7 & 5 \\ 2 & 4 & 3 \\ -1 & 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 7 & 3 \\ 2 & 4 & -1 \\ -1 & 0 & 4 \end{bmatrix}$

$AB = C$

$C \Rightarrow A(C_3 \to C_3 - 2C_1)$

∴ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $C_3 \to C_3 - 2C_1$

$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B$

$\boxed{\text{every elimentary column transformation is nothing but the corresponding elementary matrix multiplied by the given matrix.}}$

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Practice Qu

1) $A = \begin{bmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{bmatrix}$ rank $=1$ then $p = ?$

3 so that 3×3 vanish
2×2 vanish.

$\begin{bmatrix} 3 & p & p \\ p-3 & 3-p & 0 \\ p-3 & 0 & 3-p \end{bmatrix}$ two row zero $\Rightarrow$ $p-3=0$, $3-p=0$, $\underline{p=3}$.

---

(2) $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 3 \\ 1 & 3 & -1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ then $|AB - BA| =$

$|AB - BA| = |AB| - |BA| = |A||B| - |B||A|$

$|A| = (3-9)2 + (1(-3)) = -12 + 3 = \underline{-9}$

$|B| = 1(-1) = -1$

A & B are symmetric matrixes then
AB + BA is symmetric
AB − BA is skew symmetric

$\therefore$ odd order skew symmetric matrix det $=0$ Hence $|AB - BA| = 0$

$(AB-BA)^T = AB^T - BA^T$
$= B^TA^T - A^TB^T$
$= BA - AB$

$(AB-BA)^T = -(AB-BA)$

$\Rightarrow (AB-BA) =$ skew symmetric.

---

(3) if $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ & $B = A^{-1}$ then the element in the second row third column of B.

$\Rightarrow$ 3rd row second colown of $\dfrac{\text{Cofactor}}{|A|} = \dfrac{-1}{|A|} = \dfrac{-1}{2}$

option c

$|A| = 1(1) + 1(1) = 2$

4.) $A_{n\times n}$ = upper triangular matrix
such that $a_{ii} = 0$, $i = 1, 2 \dots n$
then Rank of $A^n = \underline{?}$

$$A_{(n\times n)} = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}$$

$A^n = 0$ $\therefore$ rank $= 0$

As per given conditions A is nilpotent matrix [eigen values are zeroes, trace $= 0$, $|A| = 0$]

5) If $\text{Adj } A = \begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}$ the $\text{abs}(|A|) =$

$|\text{Adj } A| = |A|^{n-1}$

$|\text{Adj } A| \Rightarrow -18(-112+20) + 11(-16+16) - 10(10-56)$

$\Rightarrow -18(-92) + 460$

$|\text{Adj } A| \Rightarrow 2116$

$\rightarrow |A|^2 = 2116$

$|A| = \pm 46$

$\boxed{\text{abs } |A| = \underline{46}}$

6) A & B symmetric $\Rightarrow$ $AB + BA$ is symmetric $\therefore$ Singular
$AB - BA$ is skew symmetric

$AB \neq BA$ $\therefore$ b)

7.) total 2 possibility for each entry $\therefore$ $\underline{2^{\frac{n(n+1)}{2}}}$
Symmetric has $\frac{n(n+1)}{2}$ independent entry

8) A is square Matrix of order $(n-1)$ then $|A| = ?$
such that $a_{ij} = \begin{cases} (n-1) & i = j \\ -1 & i \neq j \end{cases}$

$n = 2$
$A = [1]$
$\Rightarrow |A| = 1$

$A = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$ $|A| - 4 - 1 = \underline{3} =$

$n = 3$

$\therefore$ option b.

use row transformation and reduction

Linear Independent Vectors

→ The set of Vectors $\{x_1, x_2, x_3 \dots x_n\}$ is said to be linear independent vector set if there is no nonzero scalar such that its linear combination of vectors is zero.

ie) $k_1x_1 + k_2x_2 + k_3x_3 + \dots\dots + k_nx_n = 0$ linear combination.

then $\boxed{k_1 = k_2 = k_3 = \dots k_n = 0}$ linear independent.

if any $k \neq 0$ (any) then dependent.

→ Linear dependent Vectors

The set $x_1, x_2, x_3, \dots x_n$ are linearly dependent Vectors if there exist a non zero scalar $k_i \neq 0$ such that $k_1x_1 + k_2x_2 + k_3x_3 + \dots k_nx_n = 0$.

One vector can be expressed as linear combination of other vectors.

Note.

→ if A is a square matrix of order n and $|A| = 0$ then all the rows and columns are linearly dependent.

→ if $|A| \neq 0$. Then all the rows and columns are linearly independent.

Note:
- Subset of Independent set of Vectors are definitely independent
- Subset of dependent set of Vectors may be dependent or independent.
- Super set of dependent set of Vectors is also dependent.
- Super set of independent set of Vectors may or may not be independent.

10) $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ $a, b, c \neq 0$ (real no.)

then Rank of $(A) = \underline{?}$

$|A| = 0$ (Skew symmetric) with order $= 3$ (odd)

2x2 Minor of $A = \begin{vmatrix} -a & 0 \\ -b & -c \end{vmatrix} = ac \neq 0$

then rank $= 2$. option c.

if condition $a, b, c \neq 0$ is not specified

possible rank $= 0$ (for $a, b, c = 0$ all zero)

rank $= 2$ (for atleast one non zero)

→ For symmetric & skew symmetry ~~the RANK is~~ of 3rd order RANK of 1 is not possible.

→ For skew symmetric matrix 3rd order is not possible

System of Linear Equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \quad (1)$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_n + \cdots + a_{nn}x_n = b_n$$

is called system of linear equations in n-variables.

This can be expressed as matrix equations $AX = B$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ Coefficient Matrix (A)

$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ Constant matrix.

$\boxed{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ Variable matrix or Unknown matrix.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$ } System of Linear Equation in one Variable

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix}$$ Augmented Matrix AB

→ If $B \neq 0$, then $AX = B$ is known as Non-homogenous system of equ-s.

→ If $B = 0$, then $AX = 0$ is known as Homogeneous sys. of Equ-s

→ If the system has solution, then it is called consistent system.

→ If the system has no solution, then it is known as inconsistent sys.

→ The zero solution $(x_1, x_2, x_3 = 0)$ is called trivial solution.

→ A non zero solution is called non-trivial solution.

Non-Homogeneous System

$AX = B$

① If $\rho(A) \neq \rho(AB)$ then system is inconsistent. No solution

② If $\rho(A) = \rho(AB)$ then system is consistent solution exists.

consistent

$\boxed{\rho(A) = \rho(AB)}$

$\boxed{n = \text{no. of Variables}}$

n is not matrix order but no. of ~~equations~~ variables.

$\boxed{\rho(A) = \rho(AB) = n}$

unique solution.

$\boxed{\text{If } \rho(A) = \rho(AB) < n}$

Infinite no. of Solutions.

If system has unique solution then the no. of independent solution is zero. $n - \rho = n - n = 0$

No. of independent solution

- $\rho(A) = \rho(AB) = \rho$
- $(n - \rho)$ independent solution
- ∞ dependent solutions

Homogenous System $(AX = 0)$

- It is always consistent system, since it has zero solution (trivial solution).

$\rho(A)$ always $= \rho(AB)$ $B = 0$ Matrix

$\boxed{n - \text{no. of variables}}$

① $\boxed{\rho(A) = n}$ only trivial solution

② $\boxed{\rho(A) < n;\ \rho(A) = \rho}$ Infinite no. of solutions. Including both trivial & Non trivial Solutions.

If A is a square matrix of order $n \times n$

I) if $\rho(A) = n$ ie A is non singular $|A| \neq 0$ then it has unique solution which is trivial solution.

II) if $|A| = 0$ then system has infinite no. of solutions.

11) $x - 2y + z = 3$
$2x + \alpha z = -2$
$-2x + 2y + \alpha z = -1$

$\rho(A) = \rho(AB) = 3$

$\Rightarrow |A| \neq 0$

~~$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & \alpha \\ -2 & 2 & \alpha \end{bmatrix}$~~

~~$|A| = \alpha^2 + 4 + 2(2\alpha - 2) + 1(4 + 2\alpha)$~~

~~$\Rightarrow |A| = \alpha^2 + 4 + 4\alpha - 4 + 4 + 2\alpha$~~

~~$|A| = \alpha^2 + 6\alpha + 4 \neq 0$~~

$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & \alpha \\ -2 & 2 & \alpha \end{bmatrix}$

$|A| = 1(-2\alpha) + 2(2\alpha + 2\alpha) + 1(4) \neq 0$

$\Rightarrow -2\alpha + 8\alpha + 4 \neq 0$

$6\alpha + 4 \neq 0$

$\alpha \neq \frac{-4}{6} \Rightarrow \alpha \neq -2/3$

option a

12) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 4 & 3 & 10 \end{bmatrix}$

$|A| = 1(-10-12) - 2(4+4) + 3(6+4)$

$|A| = -22 - 8 + 30 = 0$

Rank = 2

A^{-1} doesn't exist

$|A| = 0 \Rightarrow$ Infinite Solution

$\therefore$ option c

13) $AB = \begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix}$

symmetric:

$|A| = k(k^2 - 1) - 1(k-1) + 1(1-k)$

$-2k + 2 + k^3 - k \neq 0$

$-3k + 2 + k^3 \neq 0$ $k \neq 1$

$k \neq -2$

option b

14) $[A|B] = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 6 & 3 & 4 & 7 \\ 4 & 2 & 1 & 3 \end{bmatrix}$ $R_3 \to R_3 - 2R_1$ $\begin{bmatrix} 2 & 1 & 0 & 1 \\ 6 & 3 & 4 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $R_2 \to R_2 - 3R_1$

$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $R_2 \leftrightarrow R_3$, $R_3 \to R_3 - 4R_2$ $\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ 1 non zero row

$\Rightarrow$ 1 dependent vector

$\Rightarrow$ 2 independent

Rank $= 2$.

$\therefore$ Infinite solution for ($AX = B$ & $AX = 0$)

$\therefore$ option c.

15.) $x + y + z = 3$
$x + 2y + 3z = 4$
$x + 4y + kz = 6$

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{vmatrix} = 0 \Rightarrow 1(2k - 12) - 1(k - 3) + 1(4 - 2) = 2k - 12 - k + 3 + 2 = 0$

$\Rightarrow \cancel{2k}\ k - 7 = 0$

$k = 7$

16.) $A = \begin{bmatrix} k & k & k \\ 0 & k-1 & k-1 \\ 0 & 0 & k^2-1 \end{bmatrix}$ if $AX = 0$ has only one independent solution, then $k = ?$

1 independent solution $\Rightarrow n - r = 1$

$r = n - 1 = 2 \quad \therefore |A| = 0$

Rank $\left(\begin{bmatrix} k & 0 & k \\ 0 & 0 & k-1 \\ 0 & 1-k^2 & k^2-1 \end{bmatrix} \right.$ $\begin{bmatrix} k & 0 & 0 \\ 0 & 0 & k-1 \\ 0 & (1-k^2) & (k^2-1) \end{bmatrix}$

$k = -1, 0 \to R_1$ all zero.

$\downarrow$

R_2 all zero

if $k = 1$ R_2, R_3 zero.

ALITER.

$|A| = k(k-1)(k^2-1) = 0 \Rightarrow k = 0$

$\Rightarrow k = 0$
$k = 1 \longrightarrow$ Not possible because
$k^2 = 1$
$k = \pm 1$

$\therefore$ $k = 0, -1$ for Rank $A = 2$

option a

If no. of eq variable less than no. of solution need not be infinite solution they may be inconsistent

17) $AX = B$

A has linearly independent column

B is a linear combination of column of A

$\rightarrow$ Rank $A = 3$ $|A| = 0$ because of independent column

$\rightarrow$ Rank of $A|B$ is 3 [no matter B] because minor of $A|B$ i.e. A (non Vanishing)

$\therefore$ $\rho(A) = \rho(A|B) = 3 = n$ $\therefore$ Unique Solution. option a

18.) $3x + 2y = 1$

$4x + 7z = 1$

$x + y + z = 3$

$x - 2y + 7z = 0$

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 4 & 0 & 7 \\ 1 & 1 & 1 \\ 1 & -2 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}$$

$$[A|B] = \begin{bmatrix} 3 & 2 & 0 & 1 \\ 4 & 0 & 7 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & -2 & 7 & 0 \end{bmatrix} \quad R_3 \leftrightarrow R_1 \begin{bmatrix} 1 & 1 & 1 & 3 \\ 4 & 0 & 7 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & -2 & 7 & 0 \end{bmatrix} \quad R_4 - R_1 \begin{bmatrix} 1 & 1 & 1 & 3 \\ 4 & 0 & 7 & 1 \\ 3 & 2 & 0 & 1 \\ 0 & -3 & 6 & -3 \end{bmatrix}$$

$R_3 - 3R_1$
$R_2 - 4R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -4 & 3 & -11 \\ 0 & -1 & -3 & -8 \\ 0 & -3 & 6 & -3 \end{bmatrix}$$

~~$R_3 \to 4R_3 - R_2$~~
~~$R_4 \to 4R_4 - 3R_2$~~

$$\cancel{\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -4 & 3 & -11 \\ 0 & 0 & -2 & -5 \\ 0 & 0 & 9 & 6 \end{bmatrix}}$$

~~$R_4 \to 2R_4 + 9R_3$~~

$$\cancel{\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -4 & 3 & -11 \\ 0 & 0 & -2 & -5 \\ 0 & 0 & 0 & -33 \end{bmatrix}}$$

~~Rank $A|B = 4$~~

~~Rank $A = 3$~~

~~$\therefore$ No solution.~~

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & -4 & 3 & -11 \\ 0 & -3 & 6 & -3 \end{bmatrix}$$

$R_3 \to R_3 - 4R_1$
$R_4 \to R_4 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 15 & 21 \\ 0 & 0 & 15 & 21 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 15 & 21 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank $(A|B) = $ Rank $(A) = 3$

$3 =$ no. of Variables $\therefore$ Unique Solution

option c

If no. of eq less than no. of Variable need not be infinite Solution. They may be inconsistent

Eigen Values & Eigen Vectors

Let $A_{n\times n}$ be a square Matrix & $I_{n\times n}$ be unit Matrix & λ be a scalar

$\Rightarrow$ then $\boxed{A-\lambda I \text{ is called Eigen Matrix / Characteristic Matrix / Latent Matrix}}$

$\Rightarrow$ $\boxed{|A-\lambda I| \text{ is called Eigen determinant or Eigen Polynomial.}}$

$\boxed{|A-\lambda I| = 0 \text{ is called eigen equation or, characteristic equation.}}$

Eigen equation of 2x2 Matrix Directly

$$\boxed{\lambda^2 - (\text{Trace of } A)\lambda + |A| = 0}$$

Eigen equation for 3x3 Matrix

$$\boxed{\lambda^3 - (\text{Trace of } A)\lambda^2 + (M_{11}+M_{22}+M_{33})\lambda - |A| = 0}$$

Q) Eigen equation of A?

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & -1 \\ 3 & 0 & 2 \end{bmatrix}$$

$$\lambda^3 - 6\lambda^2 + (4 + (-5) + 8)\lambda - [(2\cdot 4) + 1(11) + 3(-6)]$$

$$\underline{\lambda^3 - 6\lambda^2 + 7\lambda - 1 = 0}$$

Eigen Values

The solutions or roots of an eigen equation are known as Eigen roots or characteristic roots.

eg) $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$(\lambda-5)(\lambda-1)$$

$$\lambda = \underline{5, 1}$$

Eigen Vector

An non zero vector $\boxed{X \neq 0}$ is such that $AX = \lambda X$ or $(A - \lambda I)X = 0$ is called an eigen vector of matrix A corresponding to an eigen value λ

let $B = A - \lambda I$ Eigen Matrix

then $BX = 0$ $X = 0$ always a solution

but for non zero solution $|B| = 0$

$\therefore |A - \lambda I| = 0$

which are satisfied only for Eigen values λ_1, λ_2 etc.

for eigen values there can be corresponding eigenvector.

eg.) $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ Eigen vector corresponding to 1

$\Rightarrow (A - \lambda I)X = 0$

$(A - I)X = 0$

$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 0 \Rightarrow 3x_1 + 3y_1 = 0$

$x_1 + y_1 = 0$

$\Rightarrow x_1 = -y_1$

$\therefore X_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} K \\ -K \end{bmatrix}$ 1 Independent Solution
∞ dependent Solution.

Eigen Vector corresponding to 5

$(A - 5I)X = 0$

$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \Rightarrow x_1 = 3y_1$ $X_5 = \begin{bmatrix} 3k \\ k \end{bmatrix}$ 1 Independent
∞ dependent

Algebraic Multiplicity (AM)

The number of repetition of an eigen value or order of eigen value is known as Algebraic Multiplicity of eigen value λ.

Geometric Multiplicity (GM)

The no. of linear independent eigen vectors corresponding to λ is geometric multiplicity of λ. $\boxed{G.M = n - r \text{ in } (A - \lambda I)X = 0}$

$\boxed{\text{Note: } GM \leq AM}$

If $\left.\begin{matrix} AM = 1 \\ GM = 1 \end{matrix}\right\}$ distinct eigen values $\Rightarrow$ 1 set of linearly independent / eigen value.

All distinct eigen values $\Rightarrow$ All eigen vectors are independent.

- If AM & GM of ~~any~~ eigen value is same then the matrix is diagonalisable.

Inner product of Vectors

$\boxed{X \cdot Y = X^T Y}$

Inner product of two eigen vectors X & Y is given by $X \cdot Y = X^T Y$ or $Y^T X$

eg: $x = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ $y = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ then $x \cdot y =$ $1 - 2 + 3 = 2$

Orthogonal Vectors

Two vectors X & Y are orthogonal vectors to each other if $X \cdot Y = 0$

$\boxed{\text{ie inner product is zero}}$

then $X \perp Y$

eg. $x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $y = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $X \cdot Y = -1 + 0 + 1 = 0$

X & Y are orthogonal vectors. $x \perp Y$

$\boxed{\text{If a real symmetric matrix having distinct eigen values, then its eigen vectors are pair wise orthogonal vectors.}}$

Properties of Eigen Values

1) Sum of Eigen Values = trace of A
2) Product of Eigen Values = det (A) = |A|
3) If A is singular, then any one (atleast) (eigenvalue) must be zero & vice versa
4) Eigen values of diagonal, scalar, & triangular matrices are the diagonal elements itself.
5.) Eigen values of real symmetric matrix are real numbers
6) Eigen values of skew symmetric matrix are either zeroes or purely imaginary.
7) Eigen values of a orthogonal matrix are of unit modulus (~~square of terms = 1~~)
   ie $|\lambda| = 1$ eigen value lies on unit circle.
8) Eigen values of nilpotent matrix are zeroes. (trace zero for nilpotent $|A| = 0$)
9.) Eigen Values of A & A^T are same.

If λ is eigen value of A

then

① $k\lambda$ is eigen value of kA
② λ^m is eigen value of A^m
③ $\lambda \pm k$ is eigen value of $A \pm kI$
④ $1/\lambda$ is eigen value of A^{-1}
⑤ $\frac{|A|}{\lambda}$ is eigen value of Adjoint A

Properties of Eigen Vectors

1) $X \neq 0$
2) Eigen Vectors of A & A^T are not same
3) Eigen Vectors of A, kA, A^m, A^{-1} are same
4) One eigen value corresponds to more than one eigen vectors but one eigen vector cannot corresponds to more than one eigen value.

Cayley Hamilton theorm

→ Every square matrix satisfies it's own characteristic equation.
→ ie) if λ is replaced by A in eigen equation. then it is true & satisfies.

eg:- $\lambda^3 - 3\lambda^2 + 5\lambda - 7 = 0$

$\Rightarrow A^3 - 3A^2 + 5A - 7I = 0$

Minimal Polynomial

A polynomial consisting of minimum degree which satisfies the matrix A is called minimal polynomial of A.

→ If a matrix has distinct eigen values, then characteristic polynomial & minimal polynomial are same.

→ The degree of minimal polynomial greater than or equal to the number of distinct eigen values. (factor corresponding to distinct eigen value is mandatory $(A-\lambda_0)(A-\lambda_1)(A-\lambda_2)$ $\lambda_0, \lambda_1, \lambda_2$ distinct.)

→ A minimal polynomial should satisfy all eigen values of A including A.

19) ϕ $A^{-1} \to 0, 1, 3$ dosen't exist } so can't say.
$A^{-1} \to 1, 2, 3$ exist } same for Rank A = 3

∴ option d.

20) $|A| = 8(21-16) + 6(-18+8) + 2(24-14)$

$|A| \quad 8(5) + 6(-10) + 2 \times 10$

$|A| = 40 - 60 + 20 = 0$

trace = sum of eigen = 18

product of eigen value = 0

∴ option b

$\lambda^3 - 18\lambda^2 + (5+20+20)\lambda - 0 = 0$

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\lambda(\lambda-15)(\lambda-3) = 0$

∴ 0, 15, 3 //

21.) $\begin{bmatrix} a & a \\ a & a \end{bmatrix}_{2\times 3}$

$\lambda^2 - 2a\lambda = 0$

$\lambda(\lambda - 2a)$

0, 2a

$\begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}_{3\times 3}$

$\lambda^3 - 3a\lambda^2 = 0$

$\lambda^2(\lambda - 3a)$

0, 0, 3a

$\begin{bmatrix} a & \cdots & a \\ \vdots & & \\ a & \cdots & a \end{bmatrix}_{n\times n}$

$\lambda^n - na\lambda^{n-1}$

$\lambda^{n-1}(\lambda - na)$

000 ... n time, na

$\therefore A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{n\times n} = \underbrace{0 \cdots 0}_{n \text{ time}}, n$ distint $= 0, n$.

---

22. $C.E = (\lambda-1)(\lambda-2)(\lambda-3)$

$\Rightarrow (A-I)(A-2I)(A-3I) = 0$

$(A^2 - 3A + 2I)(A-3I)$

$= A^3 - 3A^2 - 3A^2 + 9A + 2A - 6I = 0$

$\Rightarrow A^3 - 6A^2 + 11A - 6I = 0$

direlly $\boxed{\lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3 = 0}$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$A^3 - 6A^2 + 11A - 6I = 0$

$A^2 - 6A + 11I - 6A^{-1} = 0$

$6A^{-1} = \underline{A^2 - 6A + 11I}$

---

23) $\begin{bmatrix} 10 & -4 \\ 18 & -12 \end{bmatrix}$

$\lambda^2 + 2\lambda + (-120 + 72) = \lambda^2 + 2\lambda - 48 = 0$

$(\lambda + 8)(\lambda - 6) = 0$

$\lambda = \underline{-8, 6}$

$(A - \lambda I)X = 0$

$\begin{pmatrix} 18 & -4 \\ 18 & -4 \end{pmatrix} X = 0$ $\quad 18x = 4y$ $\quad 9x = 2y$ $\quad X = \begin{bmatrix} 2k \\ 9k \end{bmatrix}$ $\therefore$ option b.

24) $A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\lambda = 1, 2, -1, 0$

smallest $= -1$

$[A - \lambda I] = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$

$w = 0$

$w = 0$

$3y + w = 0 \Rightarrow y = 0$

$2x + y - z + 2w = 0$

$= 2x - z = 0$

$\therefore$ option c

25) trace $= -2 + 1 + 0 = -1$

Sum of eign value $= -6 + \lambda_3 = -1$

$\lambda_3 = -1 + 6 = 5$

$[A - \lambda I]$

$= \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$-7x + 2y - 3z = 0$
$2x - 4y - 6z = 0$
$-1x - 2y - 5z = 0$

Verify options b

26) $[2 \ -2 \ 1]^T =$ eigen vector

$A \begin{bmatrix} 8 & -6 & 2 \\ -6 & x & -4 \\ 2 & -4 & 3 \end{bmatrix}$ then $x = ?$

$(A - \lambda I)X = 0 \Rightarrow \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & x-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 0$

or use $AX = \lambda X$

$16 - 2\lambda + 12 + 2 = 0 \Rightarrow 30 = 2\lambda \Rightarrow \lambda = 15$

$-12 - 2x + 2\lambda - 4 = 0$

$2x = -16 + 30 = 14 \Rightarrow x = 7$

29.) if $(1,2,0)$ is eigen vector corresponding eigen value is ____ .

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \qquad 6-\lambda-4=0$$

$$\underline{\underline{\lambda = 2}}$$

27.) $\lambda = 1,-1,3$ trace of $A^4 - 3A^3$ is ____ .

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$$

diversion

$A^4 \Rightarrow$ eigen values are $1, 1, 81$

$A^3 \Rightarrow$ eigen values are $1, -1, 27$

$3A^3 \Rightarrow$ " " " $3, -3, 81$

trace of $(A^4 - 3A^3)$ $[(1-3)+(1+3)+0] \Rightarrow -2+4 = \underline{\underline{2}}$

28.) $$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$\lambda^3 - \lambda^2 + (-4-2+2)\lambda - (1(-4)-1(0)) = 0$

$\lambda^3 - \lambda^2 - 4\lambda + 4 = 0$

$\lambda = \underline{1, 2, -2}$ $\quad (\lambda-1)(\lambda^2-4)$

$\underline{(\lambda-1)(\lambda-2)(\lambda+2)}$

then $\lambda(A^3 - A^2 - 4A + 6I)$

$\lambda_1 = (1-1-4+6) = 2$

$\lambda_2 = (8-4-8+6) = 2$

$\lambda_3 = (-8-4+8+6) = 2$

trace of $(A^3 - A^2 - 4A + 6I) = 6$

det of $(A^3 - A^2 - 4A + 6I)$

$= 2 \times 2 \times 2 = \underline{\underline{8}}$

---

30) $A^T = -A$ (skew symmetric)

$\therefore B = (A + A^T) = \phi$

eigen value $\lambda = 0, 0, 0$ (n times)

AM $= n$

GM $= ?$

GM $= n - \rho(B - \lambda I)$

$= n - \rho(0 - 0I)$ $= n - \rho(0)$

$= n$

Note: Eigen values of zero matrix are zeroes. AM of zero = order of matrix.
GM of zero = order of matrix.

31.
$$\begin{bmatrix} d_1 & & \text{Non zero} \\ 0 & d_2 & \\ 0 & 0 & d_3 \ \ddots \\ 0 & 0 & \end{bmatrix}$$

$d_1, d_2 \ldots = 0$

$$\therefore A = \begin{bmatrix} 0 & & \text{Non zero} \\ 0 & 0 & \\ 0 & 0 & 0 \ \ddots \\ 0 & 0 & 0\ 0\ 0 \end{bmatrix}$$

nilpotent matrix

$$[I+A] = \begin{bmatrix} 1 & & & \text{Non zero} \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \\ \vdots & & \ddots & 1 \end{bmatrix}$$

$\therefore$ invertible. all other options wrong.

- singular ✗ $|A| = 1$
- skewsymmetric } Non zero
- symmetric } upper triangular matrix

32) C.E $= \lambda^3 - 12\lambda^2 + a\lambda - 32$ $(\lambda_1 = 2)$ then largest abs of eigen value is —

$8 - 48 + 2a - 32 = 0$

$8 - 80 + 2a = 0$

$2a = 72$

$a = \underline{36}$

ALITER

$\lambda_1 + \lambda_2 + \lambda_3 = \text{trace} = 12$

$\lambda_1 \lambda_2 \lambda_3 = 32$

$\Rightarrow \lambda_2 + \lambda_3 = 10$

$\lambda_2 \lambda_3 = 16$

$\Rightarrow \lambda_2, \lambda_3 = 8, 2$

$\therefore$ largest $= 8$

$\frac{16}{\lambda_3} + \lambda_3 = 10$

$16 + \lambda_3^2 - 10\lambda = 0$

$\frac{10 \pm \sqrt{100-64}}{2}$

$= \frac{10 \pm 6}{2}$

$= \frac{16}{2}$ or $\frac{4}{2}$

$= 8$ or 2

33) $|A| = 2(-3) + 1(8)$

$|A| = -6 + 8 = 2$

$\therefore$ A non singular ✓

Rank A is 3 ✓

$AX = 0$ has unique solution ✓

$|A| \neq 6$.

34)
$$\begin{bmatrix} -1 & 2 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ 2 & 1 & -1 & 3 \\ 2 & 3 & 0 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 2 & 1 & 3 \\ 2 & 1 & -1 & 3 \\ 2 & 3 & 0 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 2 & 1 & 3 \\ 0 & 1 & -1 & -3 \\ 0 & 3 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix}1&0&0&3\\0&2&1&3\\0&1&-1&-3\\0&3&0&0\end{bmatrix} \rightarrow \begin{bmatrix}1&0&0&3\\0&1&-1&-3\\0&2&1&3\\0&3&0&0\end{bmatrix} \rightarrow \begin{bmatrix}1&0&0&3\\0&1&-1&-3\\0&0&3&9\\0&0&3&9\end{bmatrix} \rightarrow \begin{bmatrix}1&0&0&3\\0&1&-1&-3\\0&0&3&9\\0&0&0&0\end{bmatrix}$$

$\therefore$ Rank = 3,

a) $|A| = 0$ ✓

b) Rank of $A = 3$ ✓

c) $AX = 0$ has infinite solution $\because$ non zero ✓

d) $AX = B$ has infinite solution ✗ optiond

Some Important Matrix Results.

Trace Properties

- $tr(A \pm B) = tr(A) \pm tr(B)$
- $tr(AB) \neq tr(A) \cdot tr(B)$
- $tr(AB) = tr(BA)$

Diagonal Matrix ~~if A, B diagonal Matrix~~

then $A \pm B$ diagonal

A^2, B^2 diagonal

A^{-1}, B^{-1} diagonal

A^T diagonal

$Adj(A), Adj B$ diagonal

Transpose properties

$\boxed{(A \pm B)^T = A^T \pm B^T}$

$(AB)^T = B^T A^T$

$\boxed{(ABCD)^T = D^T C^T B^T A^T}$

$\Rightarrow \boxed{(A^n)^T = (A^T)^n}$

Symmetric & Skew Symmetric Not Written properties

if A, B symmetric then

- $A \pm B$ symmetric
- AB & $BA \neq$ sym.
- $\boxed{AB + BA \text{ Symm.}}$
- $\boxed{AB - BA \text{ skew sym}}$
- A^k, B^k symmetric $(k \in N)$

if A & A^T are square then

- $A + A^T$ = symm.
- $A - A^T$, $A^T - A$ – skew sy
- $\boxed{AA^T, A^TA - \text{symm}}$

if A & B are skew symmetric then

- $A \pm B$ = skew sy.
- A^2, B^2, are symm. ie A^{even}
- A^3, B^3 are skew sy. ie A^{odd}

* if A & B are either symmetric or skew symmetric then $AB = BA$

* if A & B are orthogonal then AB & BA are also orthogonal.

* if A & B are

- Multiplication Associative

eg. $ABC = A(BC) = (AB)C$

- Multiplication distributive over addition
- eg. $A(B+C) = AB + AC$
- eg. $(B+C)A = BC + CA$

- if A & B are non zero then $AB = 0 \Rightarrow A$ & B singular
- if A non ~~zero~~ singular then $AB = 0 \Rightarrow B = 0$

★ The each element of a row (or column) of a determinant if expressed as a sum of 2 or more terms then the determinant can be expressed as the sum of two or more determinants

eg:- $$\begin{vmatrix}a+d_1 & b+d_2 & c+d_3\\ d & e & f\\ g & h & i\end{vmatrix} = \begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} + \begin{vmatrix}d_1 & d_2 & d_3\\ d & e & f\\ g & h & i\end{vmatrix}$$

Properties of Adjoint.

- $\boxed{A \cdot Adj(A) = adj(A) \cdot A = |A| I}$ ★

- If A is diagonal then adj(A) also diagonal.

$$\boxed{A = \begin{bmatrix} l & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & n \end{bmatrix} \quad Adj A = \begin{bmatrix} mn & 0 & 0 \\ 0 & ln & 0 \\ 0 & 0 & lm \end{bmatrix}}$$

- $\boxed{Adj(A^T) = Adj(A)^T}$ ★

$Adj(A^T) = |A^T| \cdot (A^T)^{-1}$ | Since $|A^T| = |A|$ & $(A^{-1})^T = (A^T)^{-1}$

$Adj(A^T) = |A| \cdot (A^{-1})^T$

$Adj(A^T) = Adj(A)^T$

- $\boxed{adj(AB) = adj(A) \cdot adj B}$ ★

- $A = \text{symmetric} \Rightarrow Adj\, A = \text{symmetric}$

- $\boxed{Adj(k A_{n \times n}) = k^{n-1} Adj(A_{n \times n})}$

$Adj(k A_{(n \times n)}) = |k A_{(n \times n)}| \cdot (k A_{(n \times n)})^{-1}$

$Adj\, k(A_{n \times n}) = k^n |A_{n \times n}| \cdot \frac{1}{k} (A_{n \times n})^{-1}$

Since $\boxed{|kA| = k^n |A| \;\&\; (kA)^{-1} = \frac{1}{k} A^{-1}}$

$\therefore Adj(k A_{n \times n}) = k^{n-1} Adj(A_{n \times n})$

Properties of Inverse

- $\boxed{(A^T)^{-1} = (A^{-1})^T}$
- $\boxed{(ABCD)^{-1} = D^{-1} C^{-1} B^{-1} A^{-1}}$
- $A \rightarrow \text{symmetric}$ $\boxed{A^{-1} \text{ symmetric}}$ ★
- If $AB = BA$ then $A^{-1} B^{-1} = B^{-1} A^{-1}$
- If A orthogonal then A^T & A^{-1} orthogonal
- $\boxed{kA^{-1} = \frac{1}{k} A^{-1}}$

$X \cdot Y = X^T Y = Y^T X$

$X \cdot Y = 0 \Rightarrow$ orthogonal / perpendicular Vector

$X \cdot Y = \pm 1 \Rightarrow$ parallel vector.

$\sqrt{X X^T} = \sqrt{X \cdot X} =$ (length of Vector X / norm of " X)

$\|X\| = \sqrt{X \cdot X} = \sqrt{X X^T} = 1 \Rightarrow$ normal vector

- If orthogonal each other & normal to self then orthonormal

ie $X_1, X_2, X_3 \ldots$ of same order

then $\boxed{X_i^T X_j = \begin{cases} 0, & \forall i \neq j \\ 1, & \forall i = j \end{cases}}$

- $\boxed{\text{If } \lambda \text{ is eigen value of orthogonal matrix} \Rightarrow 1/\lambda \text{ also eigen value}}$
- $\boxed{a_0 + a_1 \lambda + a_2 \lambda^2 + \ldots \text{ is an eigen value of } B = a_0 I + a_1 A + a_2 A^2 + \ldots}$
- $\boxed{\text{If matrix satisfies an equation then eigen values also satisfy}}$
- $\frac{|A|}{\lambda}$ is eigen value of $(Adj\, A)$
- If $a + \sqrt{b}$ is eigen value then $a - \sqrt{b}$ also eigen value.
- If $a + jb$ is eigen value then $(a - jb)$ also eigen value.
- $\boxed{\text{Eigen Vectors of } A \;\&\; a_0 I + a_1 A + a_2 A^2 \ldots \text{ all same.}}$

- If $X_1 X_2 X_3$ are linearly independent vectors for $\lambda_1\ \lambda_2\ \lambda_3$

then $P = [X_1\ X_2\ X_3] \Rightarrow \boxed{P^{-1}AP = D}$ $\boxed{P^{-1}A^kP = D^k}$ $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

or $\boxed{\begin{array}{l} A = PDP^{-1} \\ A^k = PD^kP^{-1} \end{array}}$

- The polynomial of lowest degree that annihilates A is called minimal poly.
- Minimal poly perfectly divides C.E polynomial
- λ eigen value satisfies both C.E & Minimal poly.
- degree of Min poly $= d$
  No. of disting eigen value $= \varepsilon$
  degree of C.E $= \epsilon$

$\boxed{\varepsilon \leq d \leq \epsilon}$

- For matrix equation post multiplication & pre multiplication must be done likewise on both sides while solving.
- To check whether vectors are linearly independent $\Rightarrow$ form matrix find rank.
- To find rank, need not always convert to row echelon. Rank property or definition can be applied (if $|A| = 0$, then rank $< n$)
- When A given and equation of A with higher power asked find C.E.
  eg $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ find A^9 (Soln $A^9 = 625A$)
- If a matrix is defined in terms of n, i, j by some equation & some property is asked it is always feasible to create a representative of the matrix by assuming values.
- zero eigen value $\Rightarrow |A| = 0$ & vice versa
- distinct eigen value $\rightarrow$ eigen vectors linearly independent
-

$\frac{11110 \times 5!}{2}$

$= \frac{1333200}{2} = 6666600$

0 0 0 0

1 3 5 7 9

9 7 5 3 1

1 1 1 1 1 0

1,1 1,1 1 0

0 0 0 0

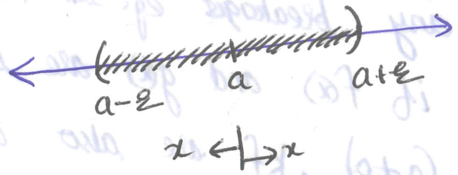
7 5 9 3 1

3 5 1 7 9

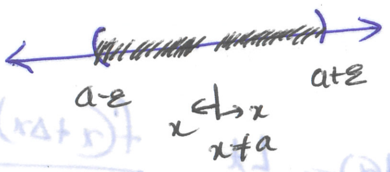
1 1 1 1 1 0

Differential Calculus

- Let $a \in \mathbb{R}$ be an real numbers. If $0 < \varepsilon < 1$ be any small positive real number then $|x - a| < \varepsilon$ represents all the real numbers in $(a-\varepsilon, a+\varepsilon)$ which is called ε neighbourhood of the point A.



- $0 < |x-a| < \varepsilon$ represents all the real numbers in the ~~open interval a-ε~~ deleted ε neighbourhood of the $(a-\varepsilon, a+\varepsilon) - \{a\}$ which is called point A.



Limit of a real function

Let $y=f(x)$ be any real function and let $a \in R$ if there exists two small positive real no. ϵ & δ such that $0<|x-a|<\delta \Rightarrow |f(x)-L|<\epsilon$ then we say that as x approaches a $f(x)$ approaches L and is denoted by

$$\underset{x\to a}{Lt}\ f(x) = L.$$

Continuity of a real function

Suppose $|x-a|<\delta \Rightarrow |f(x)-f(a)|<\epsilon$ then we write $\underset{x\to a}{\text{Limit}}\ f(x)=f(a)$. Now we say that the function $f(x)$ is continuous at $x=A$.

- Geometrically $y=f(x)$ is continous means its graph is a continous curve without any breakages eg: $\sin x, \cos x$ e^x, x^2, x^3
- We can also verify that if $f(x)$ and $g(x)$ are continous then $f+g, f-g, f\times g, f/g\ (g\neq 0)$, kf are also continous.
- We always observe that every continous function has limit value but if a function has limit value, then it may be continous or discontinous also.

Derivative of real function

Let $y=f(x) \Rightarrow \dfrac{dy}{dx} = f'(x) = \underset{\Delta x\to 0}{Lt} \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

Similarly $\left[f'(x)\right]_{@x=a} = f'(a) = \underset{x\to a}{Lt}\left[\dfrac{f(x)-f(a)}{x-a}\right]$

Note.

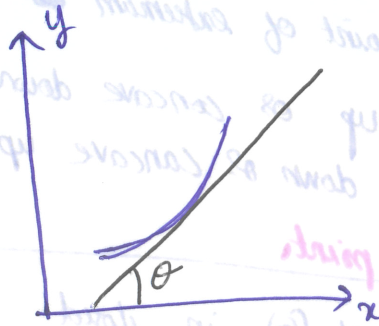
$\boxed{\text{From above equation we can observe that if a function } f(x) \text{ has derivative at } x=a \Rightarrow \text{it is also continuous @ } x=a \Rightarrow Lt \text{ also exist at } x=a.}$

Partial ~~dv~~ derivatives

Let $f(x,y)$ be an implicit function, then $\dfrac{\partial f}{\partial x} = \underset{\Delta x \to 0}{Lt} \dfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$

Similarly $\dfrac{\partial f}{\partial y} = \underset{\Delta y \to 0}{Lt} \dfrac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$

Increasing and decreasing functions



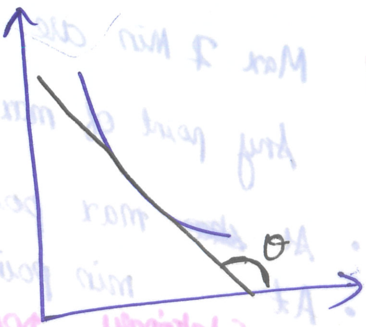
$\boxed{\dfrac{dy}{dx} > 0}$

$\dfrac{dy}{dx} = \tan\theta$

$0 < \theta < 90$

$\Rightarrow \dfrac{dy}{dx} = +ve$

$\therefore$ Increasing function



$\boxed{\dfrac{dy}{dx} < 0}$

$\dfrac{dy}{dx} = \tan\theta$

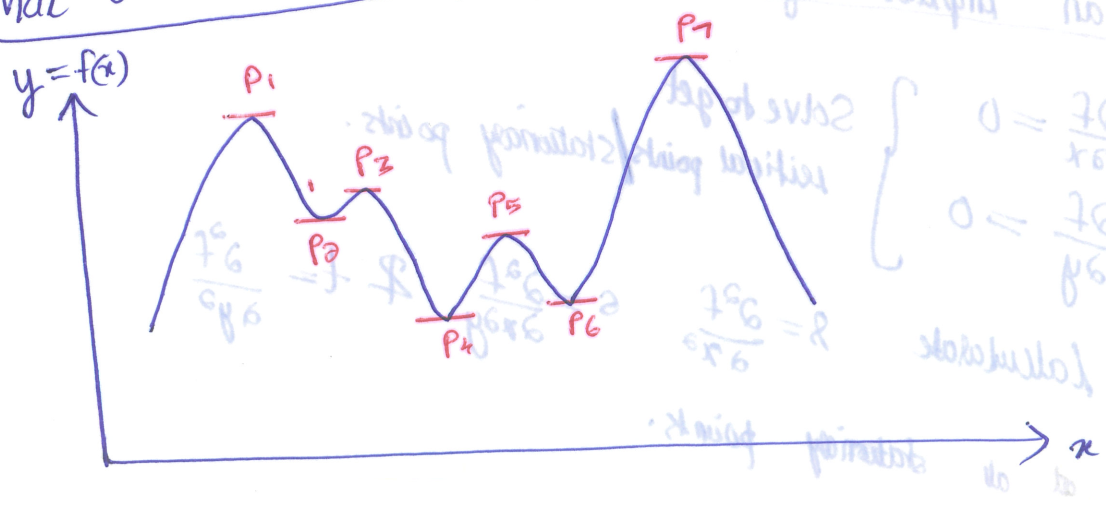
$90 < \theta < 180$

$\tan\theta = -ve$

$\dfrac{dy}{dx} = -ve$

$\therefore$ Decreasing function

Max or Min value of a function of Single Variable



- In the above graph we can observe that the function has max values at points P_1, P_3, P_5, P_7 & min values at P_2, P_4, P_6 as x is changing from a to b.
- At a point of max or min we can observe that $f'(x)=0$ that means tangents are parallel to x axis.
- ∴ By solving $f'(x)=0$ we get all stationary points.
- By solving $f''(x) < 0$ at any stationary point max value of $f(x)$ occurs at that point, similarly $f''(x) > 0$ @ any stationary point then min value of $f(x)$ occurs at that point.
- If $f''(x)=0$ it is neither maxima or minima of $f(x)$. and that point is called ~~inflection point or po~~ inflexion point.
- Max & Min are combinely called extreme value.
- Any point of max or min is called point of extremum
- At max point shape = convex up or concave down
- At min point shape = convex down or concave up.

Stationary point same as critical point.

In particular, to decide the extreme value of $f(x)$ in closed interval a, b, then we must consider, the function values at the boundaries ie $f(a)$ & $f(b)$ in addition to the stationary points.

Working Procedure of finding Max/Min value of an implicit function $f(x,y)$

I) $\left.\begin{array}{l} \dfrac{\partial f}{\partial x}=0 \\ \dfrac{\partial f}{\partial y}=0 \end{array}\right\}$ Solve to get critical points/stationary points.

II) Calculate $r=\dfrac{\partial^2 f}{\partial x^2}$ $s=\dfrac{\partial^2 f}{\partial x \partial y}$ & $t=\dfrac{\partial^2 f}{\partial y^2}$

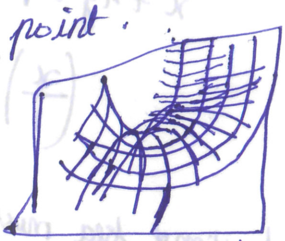
at all stationary points.

III) i) $(rt - s^2) > 0$ & $r < 0$ Stationary point is maxima point.

ii) $(rt - s^2) > 0$ & $r > 0$ Stationary point ~~...~~, ~~...~~ is minima point.

iii) $(rt - s^2) < 0 \longrightarrow$ Stationary point is saddle point.

iv) $(rt - s^2) = 0 \longrightarrow$ No conclusion can be made about the stationary point



Saddle point.

Q9. Page 129

Maxima of $f(x) = \frac{x^3}{3} - x$

$\frac{df}{dx} = x^2 - 1 = 0 \Rightarrow x = \pm 1$

$\frac{d^2f}{dx^2} = 2x$ $\therefore$ Maxima at $\underline{-1}$ Since $\left.\frac{d^2f}{dx^2}\right|_{x=-1} = -2 < 0 \Rightarrow$ Maxima.

---

30) Maximum value of function $f(x) = x^3 - 6x^2 + 9x + 1$ in $[0,2]$

$f'(x) = 3x^2 - 12x + 9 = 0$

$\Rightarrow x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0 \Rightarrow x = 1, 3$

$f''(x) = 6x - 12$

$f''(1) = -6 \rightarrow$ maxima

$f''(3) = 6 \rightarrow$ minima

$\therefore f''(1)$ is a maxima in $(0,2)$

$f(0) = 0 - 0 + 0 + 1 = 1$

$f(1) = 1 - 6 + 9 + 1 = 5$

$f(2) = 8 - 24 + 18 + 1 = 3$

$f(3) = 27 - 54 + 27 + 1$

$f(3) = 1$

---

31) Range of k for which $f(x) = (k^2-4)x^2 + 6x^3 + 8x^4$ has maxima at $x = 0$.

$f'(x) = 2(k^2-4)x + 18x^2 + 32x^3 = 0$

$f'(x) = 0 \Rightarrow (k^2-4) + 9x + 16x^2 = 0 \Rightarrow \boxed{(k^2-4) = 0 \Rightarrow k^2 = 2, \ k = \pm 2}$ X

Since x is a common factor & was taken out $x=0$ is anyway sol

$x = 0$ is maxima $\Rightarrow x = 0$ is a valid solution

$f''(x) = 2(k^2-4) + 36x + 96x^2$

$f''(x) = 2[(k^2-4) + 18x + 48x^2]$

K is max $\Rightarrow \left. f''(x)\right|_{x=0} < 0 \Rightarrow \left.(k^2 - 4 + 18x + 48x^2)\right|_{x=0} < 0$

$k^2 - 4 < 0$

$k^2 = \pm 2$ @ $k^2 - 4 = 0$

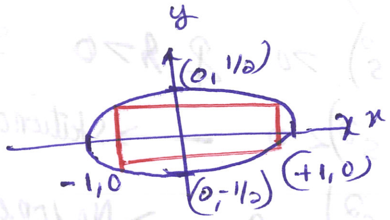
$\Rightarrow -2 < k < 2$

$k \in (-2, 2)$

Q32) Maximum Area of a rectangle whose ~~abc~~ vertices lie on ellipse

$$x^2+4y^2=1$$

$$\left(\frac{x}{1}\right)^2+\left(\frac{y}{1/2}\right)^2=1$$



function of Area must be maximised.

$$\left.\begin{array}{l}\text{length} = 2y \\ \text{width} = 2x\end{array}\right\} \text{Area} = 4xy$$

$$\left(y^2=\frac{1-x^2}{4}\right)$$

$$A^2 = 16x^2y^2$$

$$A^2 = 4x^2(1-x^2)$$

$$f(x) = -4x^4+4x^2$$

Justified as max of area and max of Area^2 is same since it is a positive function.

$$f'(x) = -16x^3+8x \Rightarrow \left(x^2=1/2\right)$$

$$f'(x)=0 \Rightarrow x=0, \pm\left(\frac{1}{\sqrt{2}}\right)$$

$$\therefore x=\frac{1}{\sqrt{2}} \quad y=\sqrt{\frac{1-1/2}{4}} = \frac{1}{2\sqrt{2}}$$

$$\text{Area} = 4xy = 4\times\frac{1}{\sqrt{2}}\times\frac{1}{2\sqrt{2}} = 1$$

If $A=4xy = 4x\sqrt{\dfrac{1-x^2}{4}}$ } difficult to maximise.

$$A = 2x\sqrt{1-x^2}$$

Q33) The maximum value of $f_{(x,y)} = x^3+3xy^2-15x^2-15y^2+72x$ is.

$$p=\frac{\partial z}{\partial x} = 3x^2+3y^2-30x+72=0$$

$$q=\frac{\partial z}{\partial y} = 6xy-30y=0$$

$$\Rightarrow x^2+y^2-10x+24=0$$

$$xy-5y=0$$

$$xy=5y=0$$

$$\Rightarrow y=0$$

$$\text{or } x=5$$

if $y=0$

$$x^2-10x+24=0 \Rightarrow x=\frac{10\pm\sqrt{100-96}}{2}$$

$$x=\frac{10\pm2}{2} = 6,4$$

if $x=5$

$$25+y^2-50+24=0$$

$$y^2=1$$

$$y=\pm1$$

$\therefore$ Stationary points.

$(6,0), (4,0), (5,1), (5,-1)$

$r = \frac{\partial^2 f}{\partial x^2} = 6x - 30$

$s = \frac{\partial^2 f}{\partial x \partial y} = 6y$

$t = \frac{\partial^2 f}{\partial y^2} = 6x - 30$

$\therefore (x_1, y_1) = (4, 0)$ Maxima

$(6, 0)$ Minima.

Maxima value $= f(4, 0)$

$= 64 - 240 + 288 = 112$

$rt - s^2 = (6x-30)^2 - 36y^2$

$(rt-s^2)\big\vert_{(x,y)=(6,0)} = (36-30)^2 - 0 = 36$	$r = 36 - 30 = 6 > 0$ $\therefore$ minima	
$(rt-s^2)\big\vert_{(x,y)=(4,0)} = (24-30)^2 - 0 = 36$	$r = 24 - 30 = -6 < 0$ $\therefore$ maxima.	
$(rt-s^2)\big\vert_{(x,y)=(5,1)} = -36 < 0$	Saddle point	
$(rt-s^2)\big\vert_{(x,y)=(5,-1)} = -36 < 0$ saddlepoint		

34) Maximum value of the determinant amoung all 2×2 real symmetric matrices with trace 10 is ______.

$$\begin{vmatrix} a & b \\ b & (10-a) \end{vmatrix} = (10-a)a - b^2$$

$f_a(a,b) = 10 - 2a = 0 \Rightarrow a = 5$

$f_b(a,b) = -2b = 0 \Rightarrow b = 0$

$$\begin{vmatrix} 5 & 0 \\ 0 & 5 \end{vmatrix} = 25.$$

24.) If $u = x^3 + xz^2 + y^3 + xyz$ where $x = e^t$, $y = \cos t$, $z = t^3$ then $\frac{du}{dt}$ @ $t = 0$ is

$$\frac{du}{dt} = \left(\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t}\right) + \left(\frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t}\right) + \left(\frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial t}\right)$$ Total derivative.

$$\frac{du}{dt} = \left((3x^2 + z^2 + yz)e^t\right) + \left((3y^2 + xz) \cdot -\sin t\right) + \left((2xz + xy) \cdot 3t^2\right)$$

@ $t = 0$

$x = e^0 = 1$

$y = \cos 0 = 1$

$z = t^3 = 0$

$$\left.\frac{du}{dt}\right|_{t=0} = \left[3e^t + 3(-\sin t) + 3t^2\right] = 3$$

25) $x^y + y^x + c = 0$ then $\frac{dy}{dx}$ @ $(1,1)$

put $f = x^y + y^x + c = 0$

$f(x,y) = c$ is an implicit function, then

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

$$\frac{dy}{dx} = \frac{-\left(yx^{y-1} + y^x \log y\right)}{\left(x^y \log x + xy^{x-1}\right)}$$

Then $\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} = \frac{-(1+0)}{(0+1)} = -1$

26.) if $u = x\log(xy)$ where $x^3+y^3+3xy = 1$

$$\frac{du}{dx} = \log(xy) + \frac{x}{xy}\cdot\left(y + x\frac{dy}{dx}\right)$$

$$x^3+y^3+3xy=1$$

$$3x^2 dx + 3y^2 dy + 3x dy + 3y dx = 0$$

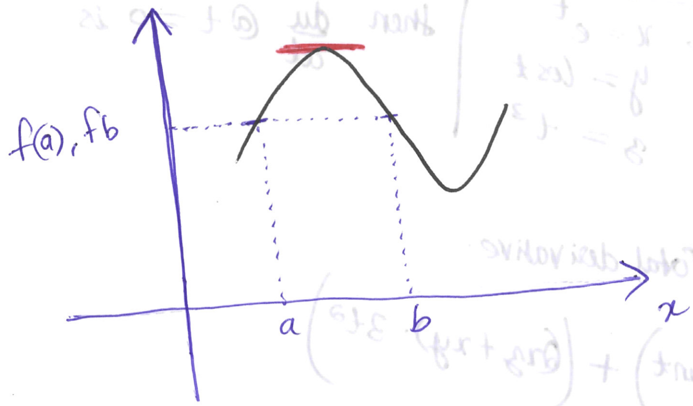
$$(3x^2+3y)\,dx + (3y^2+3x)\,dy$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(3x^2+3y)}{(3y^2+3x)}$$

$$\Rightarrow \frac{du}{dx} = \log(xy) + 1 - \frac{x}{y}\left(\frac{3x^2+3y}{3y^2+3x}\right)$$

$$\frac{du}{dx} = \log(xy) + 1 - \frac{x}{y}\left(\frac{x^2+y}{y^2+x}\right)$$

Mean value theorm



Rolle's theorm

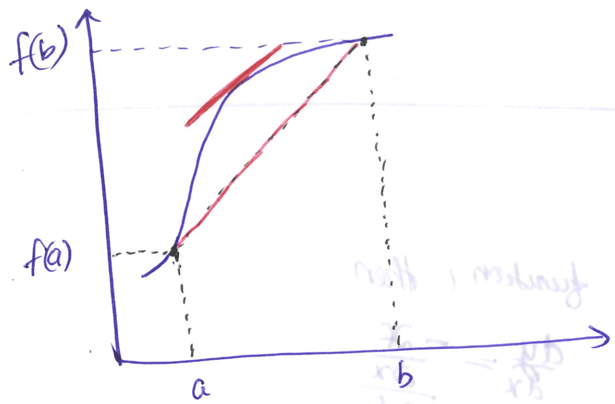
i) $f(x)$ is continous in $[a,b]$

ii) $f'(x)$ exists in (a,b)

iii) $f(a) = f(b)$

Then there is atleast one pt c such that $c \in (a,b)$ & $f'(c) = 0$

Lagrange's Theorm:-



i) $f(x)$ is continous in $[a,b]$

ii) $f'(x)$ exists in (a,b)

then there exist atleast one point c such that $c \in (a,b)$ & $f'(c) = \frac{f(b)-f(a)}{b-a}$

In the above statement if we consider $f(b)=f(a)$, then the Lagrange's theorem reduces to $f'(c)=0$. ie) Rolle's theorem is obtained from Lagrange's theorem.

Cauchy's theorem

the condition $g(b)-g(a)\neq 0$ might not cover $g'(c)\neq 0$ but $g'(c)\neq 0$ cover both

i) $f(x)$ & $g(x)$ are continous in $[a,b]$
ii) $f'(x)$ & $g'(x)$ exist in (a,b)
iii) $g'(x)\neq 0\ \forall\ x\in(a,b)$

Then there exist atleast on point $c\in(a,b)$ such that $\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$

- In the above statement suppose $g(x)=x+1\ \forall x\in(a,b)$ then equation above reduces to equation 2. ie) Lagrange's theorem is obtained from Cauchy's theorem.
- Similarly, by considering $f(x)=k$ & $g(x)=x+1\ \forall x\in(a,b)$ then above equation reduces to rolles theorem. ie) Rolle's theorem can also be obtained from cauchy's theorem.

Taylor's & Maclaurin's Series

If $f(x)$ has derivatives of every order at a point $x=a$ then we can write $\boxed{f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\cdots}$

Which is called Taylor Series expansion of $f(x)$ @ $x=a$

In the above statement put $a=0$, we get $\boxed{f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots}$

Which is called Maclaurin's Series expansion of $f(x)$ @ $x=0$.

Note:
From the Maclaurin series expansion, we can have expansions of e^x, $\sin x$, $\cos x$, $\log(1+x)$, binomial expansion etc.

Evaluation of Limits in indeterminate form.

$\frac{0}{0}$

$$\underset{x\to a}{Lt}\ \frac{f(x)}{g(x)} = \underset{x\to a}{Lt}\ \frac{f'(x)}{g'(x)} \quad \left[\text{if } f(a) = g(a) = 0\right]$$

$$\underset{x\to a}{Lt}\ \frac{f'(x)}{g'(x)} = \underset{x\to a}{Lt}\ \frac{f''(x)}{g''(x)} \quad \left[\text{if } f'(a) = g'(a) = 0\right]$$

$\frac{\infty}{\infty}$

$$\underset{x\to a}{Lt}\ \frac{f(x)}{g(x)} = \underset{x\to a}{Lt}\ \frac{f'(x)}{g'(x)} \quad \left[\text{if } f(a) = g(a) = \infty\right]$$

$$\underset{x\to a}{Lt}\ \frac{f'(x)}{g'(x)} = \underset{x\to a}{Lt}\ \frac{f''(x)}{g''(x)} \quad \left[\text{if } f'(a) = g'(a) = \infty\right]$$

$0 \cdot \infty$ form $\quad \boxed{f(a)=0 \quad g(a)=\infty}$

$$\underset{x\to a}{Lt}\ f(x)\cdot g(x) = \underset{x\to a}{Lt}\ \frac{f(x)}{\frac{1}{g(x)}} \quad \left(\frac{0}{0}\right)$$

$$\underset{x\to a}{Lt}\ f(x)\cdot g(x) = \underset{x\to a}{Lt}\ \frac{g x}{\frac{1}{f(x)}} \quad \left(\frac{\infty}{\infty}\right)$$

$\infty - \infty$ form

$$\underset{x\to a}{Lt}\ f(x) - g(x) \quad (\infty - \infty \text{ form}) \quad \left[\begin{matrix} f(a) = \infty \\ g(a) = \infty \end{matrix}\right]$$

This can also be reduced to 0/0 or $\frac{\infty}{\infty}$ form or 0∞ form. Hence estimation of limit is possible after conversion to earlier form.

$1^\infty, \infty^0, 0^0$ --- forms.

$$\underset{x\to a}{Lt}\ f(x)^{g(x)} = K$$

$$\ln K = \underset{x\to a}{Lt}\ g(x) \cdot \ln f(x)$$ will be reduced to 0 to ∞ form.

then $k = e^{\ln k}$ after finding value of $\ln k$.

1) Estimate $\ln k$ first

2) $k = e^{\ln k}$.

Q15 Page 127

$f(x) = x + 4/x$ $[1,8]$ is continous in $[a,b]$

$f(1) = 1+4 = 5$

$f(8) = \frac{17}{2}$

$f'(c) = \frac{\frac{17}{2} - 5}{7} = \frac{17-10}{14} = \frac{7}{14} = 0.5$

$f'(x) = 1 + \frac{-4}{x^2} = 0.5 \Rightarrow \frac{4}{x^2} = 0.5$

$x^2 = 8$ $\quad x = \underline{\underline{2\sqrt{2}}}$.

$f(x)$ continous in (a,b) $\quad \boxed{\mp 2\sqrt{2} \text{ not in range}}$.

Q16)

$f(0) = 0$ $\quad f'(x) = \frac{1}{1+x^2}$

$f(2) = ?$

$f'(c) = \frac{1}{1+c^2} = \frac{f(2) - f(0)}{(2-0)} \Rightarrow \frac{2}{1+c^2} = f(2)$

for $c \in (0,2)$ $\quad \underline{f(2) \in (0.4, 2)}$ [According to Lag. theorem there must be a c in that range which satisfies eq.]

Q17) $f(2) = e^2$ | $g(2) = e^{-2}$

$f(3) = e^3$ | $g(3) = e^{-3}$

$\frac{f'(c)}{g'(c)} = \frac{e^3 - e^2}{e^{-3} - e^{-2}}$

- both $f(x)$ & $g(x)$ are continous in any interval
- both $f'(x)$ & $g'(x)$ are continous in any interval.
- ..

$f'(x) = e^x$

$g'(x) = -e^{-x}$

$g'(x) \neq 0$ in $x \in (2,3)$

$\Rightarrow \frac{f'(x)}{g'(x)} = \frac{e^x}{-e^{-x}} = \frac{e^3 - e^2}{e^{-3} - e^{-2}}$

$-e^{2x} = \frac{e^2(e-1)}{e^{-3}(1-e)} = -e^5$

$\Rightarrow 2x = 5 \Rightarrow x = 5/2$

Q18) $f(x) = e^{\sin x}$ about $x=0$. Taylor Series $= ?$

$\Rightarrow f(x)_{@x=0} = \underline{\underline{1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}}}$

$f(0) = 1$

$f'(0) = e^{\sin x} \cdot \ln e = e^{\sin x} \cos x \Big|_0 = 1$

$f''(0) = e^{\sin x} \cdot \cos x - \sin x e^{\sin x} = 1$

$f'''(0) = 1$

19.) $f(x) = \log \sec x$ about $x^4 =$ ____.

Coefficient of $x^4 = \frac{f^{IV}(0)}{4!}$

$f'(x) = \frac{1}{\sec x} \cdot \sec x \tan x$

$f''(x) = \sec^2 x$

$f'''(x) = 2\sec x \cdot \sec x \tan x = 2\sec^2 x \tan x$

$f'''(x) = 2\sec x \cdot \sec x \tan x$

$f^{IV}(x) = 4\sec x \cdot \sec x \tan x \cdot \tan x + \sec^2 x \cdot 2\sec^2 x$

$f^{IV}(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$

$f^{IV}(0) = 4(1)^2 \cdot (0)^2 + 2(1)^4 = 2$

$\therefore f^{(IV)} = \frac{2}{24} = \frac{1}{12}$

---

20) $f(x) = \tan^{-1}(x)$ @ $x = 0$ series $= ?$

$f'(x) = \frac{1}{1+x^2}$

$f''(x) = \frac{-1}{(1+x^2)^2} \cdot 2x$

$f'''(x) = -\left[\frac{(1+x^2)2 - 2x[2(1+x^2)\cdot 2x]}{(1+x)^4}\right]$

$f'(0) = 1$ $\quad f'''(0) = \frac{-2}{1} = -2$

$f''(0) = 0$

$f(x) = x - \frac{2x^3}{3!} + \dots = \underline{x - \frac{x^3}{3}}$

21.) $f(x) = e^{x+x^2}$ $\quad f(0) = 1$

$f'(x) = e^{x+x^2} \cdot (1+2x) \Rightarrow f'(0) = 1$

$f''(x) = e^{x+x^2}(1+2x)^2 + 2e^{x+x^2} = e^{x+x^2}\left[\cancel{1+2x+2+[2x(1+2x)]}\right] = \cancel{e^{x+x^2}}\left[1+2x+2x+2+4x^2\right]$

$= e^{x+x^2}(2+1+4x^2+4x)$

$f'''(x) = e^{x+x^2} \cdot (1+2x)(3+4x+4x^2) +$

$e^{x+x^2}[8x+4]$

$f''(0) = 3$

$f'''(0) = 4+3 = 7$

$\Rightarrow f(x) = 1 + x + \frac{3x^2}{2!} + \frac{7x^3}{3!}$

$f(x) = \underline{1 + x + \frac{3x^2}{2} + \frac{7x^3}{6}}$

$f(x) = e^{x+x^2} = e^u$

$x+x^2$ @ $x=0 = 0$

$(x+x^2) = u$

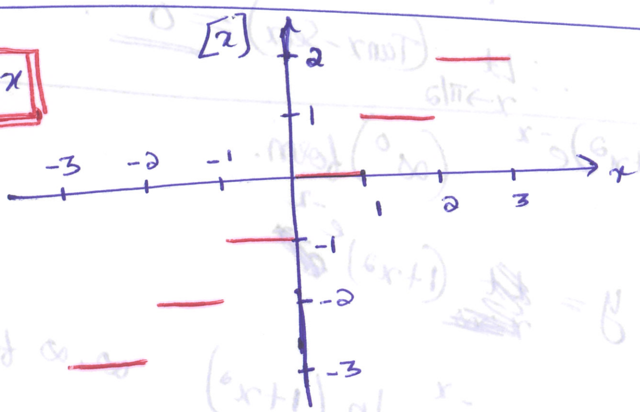
e^u expansion about $0 = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!}$

---

1) $\boxed{[x] \rightarrow \text{greatest integer less than } x}$

eg $[1.5] = 1$

$[-1.5] = -2$



$[x]$ = step function.

$\lim\limits_{x \to 5/4} (x - [x]) = 1.25 - 1 = \underline{1/4}.$

left limit = right limit

check if both side limit equal.

$\lim\limits_{x \to 5/4^+} f(x) = \lim\limits_{x \to 5/4^-} f(x)$ then limit exist.

---

2) $\lim\limits_{x \to 2} \frac{|x-2|}{x-2} = ?$ where $|x-2|$ is modulus function.

$\left(\lim\limits_{x \to 2^+} = 1\right) \neq \left(\lim\limits_{x \to 2^-} = -1\right)$

left hand limit not equal to right hand limit

$\therefore$ Doesnot exist.

---

3) $\lim\limits_{x \to 4} [x] =$ ______ $[x] \rightarrow$ step function.

$\left(\lim\limits_{x \to 4^+} [x] = 4\right) \neq \left(\lim\limits_{x \to 4^-} [x] = 3\right)$

left hand limit $\neq$ Right hand limit

$\therefore$ Does not exist.

4.) $\lim\limits_{x \to 0} \left(\frac{x \sin x}{1 - \cos x}\right)$ 0/0 form. $\frac{x\cos x + \sin x}{\sin x}$ (0/0) $\Rightarrow \frac{-x\sin x + \cos x + \cos x}{\cos x}$

$\lim\limits_{x \to 0} \left(\frac{x \sin x}{1 - \cos x}\right) = \underline{2}$

5.) $\underset{x\to\pi/2}{Lt}\ \underline{\tan x - \sec x}$ $(\infty-\infty)$ form.

$$= \underset{x\to\pi/2}{Lt}\left(\frac{\sin x}{\cos x} - \frac{1}{\cos x}\right) \quad \underset{x\to\pi/2}{Lt}\left(\frac{\sin x - 1}{\cos x}\right) \quad (0/0) \text{ form.}$$

$$\Rightarrow \left.\frac{\cos x}{-\sin x}\right|_{x=\pi/2} = \frac{0}{-1} = 0$$

$$\therefore \underset{x\to\pi/2}{Lt}(\tan x - \sec x) = \underline{\underline{0}}.$$

6.) $\underset{x\to\infty}{Lt}\ (1+x^2)^{e^{-x}}$ (∞^0) form.

$$y = \cancel{Lt}\ (1+x^2)^{e^{-x}}$$

$$\ln y = e^{-x} \cdot \ln(1+x^2) \quad \infty \cdot \infty \text{ form}$$

$$\ln y = \frac{\ln(1+x^2)}{e^x} \quad \frac{\infty}{\infty} \text{ form}$$

$$\underset{x\to\infty}{Lt}\ \ln y = \frac{\frac{1\cdot 2x}{1+x^2}}{e^x} = \frac{2x}{(1+x^2)e^x} = \frac{2}{e^x(1+x^2) + 2xe^x}$$

$$\Rightarrow \underset{x\to\infty}{Lt}\ \ln y = \frac{2}{\infty} = 0 \Rightarrow \underset{x\to\infty}{Lt}\ y = e^0 = \underline{\underline{1}}.$$

7.) $\underset{x\to 0}{Lt}\ x^x$ (0^0) form

$$x^x = y \Rightarrow \ln y = x\ln x \Rightarrow \underset{x\to 0}{Lt}\ \ln y = x\ln x = (0\cdot\infty) \text{ form.}$$

$$= \underset{x\to 0}{Lt}\ \ln(y) = \frac{\ln x}{1/x} \quad (\infty/\infty) \text{ form} = \frac{1/x}{-1/x^2} = -x = 0$$

$$\Rightarrow y = e^0 = 1$$

8) $\underset{x\to 0}{Lt} \dfrac{\sqrt{a+x} - \sqrt{a-x}}{x}$ (0/0) form.

$\dfrac{d(x^{1/2})}{dx} \quad \dfrac{+1}{2}\, x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

$\underset{x\to 0}{Lt} \dfrac{\dfrac{+1}{2\sqrt{a+x}} + \dfrac{1}{2\sqrt{a-x}}}{1} = \dfrac{+1}{\sqrt{a}}$

$\therefore \boxed{\dfrac{d(\sqrt{x})}{dx} = \dfrac{1}{2\sqrt{x}}}$

9.) $\boxed{\text{Continous} \Rightarrow \left[\underbrace{\underset{x\to a^+}{Lt} f(x) = \underset{x\to a^-}{Lt} f(x) = \underset{x\to a}{Lt} f(x)}_{\text{Limit exist}}\right] = f(a)}$

and its equal to function value.

a) $f(x) = \begin{cases} 3 & x=2 \\ 2x-1 & x>2 \\ \frac{x+7}{3}, & x<2 \end{cases}$ $\quad \left.\begin{array}{l} f(2^+) = 3 \\ f(2^-) = 3 \end{array}\right\}$ Limit exist & equal to f(a).

$f(2) = 3$

b) $f(x) = \begin{cases} 2 & x=2 \\ (8-x), & x\neq 2 \end{cases}$ $\quad \left.\begin{array}{l} f(2) = 2 \\ f(2^+) = f(2^-) = 6 \end{array}\right\}$ not continous $\left\{\begin{array}{l}\text{limit value not equal to} \\ \text{function value}\end{array}\right\}$

c) $f(x) = \begin{cases} x+2 & x\leq 2 \\ x-4 & x>2 \end{cases}$ $\quad \left.\begin{array}{l} f(2) = f(2^-) = 4 \\ f(2^+) = -2 \end{array}\right\}$ not continous

d) $f(x) = \dfrac{1}{x^3-8} \quad x\neq 2$ $\quad \therefore$ Discontinous $\quad \therefore$ only option a is continous

function not defined @ 2

10.) $g(x) = \begin{cases} -x & x\leq 1 \\ x+1 & x>1 \end{cases}$ $\qquad f(g(x)) = \begin{cases} 1+x & x\leq 0 \\ x^2 & 0<x<1 \\ (x+1)^2 & x>1 \end{cases}$

★ $f(x) = \begin{cases} 1-x & x\leq 0 \\ x^2 & x>0 \end{cases}$

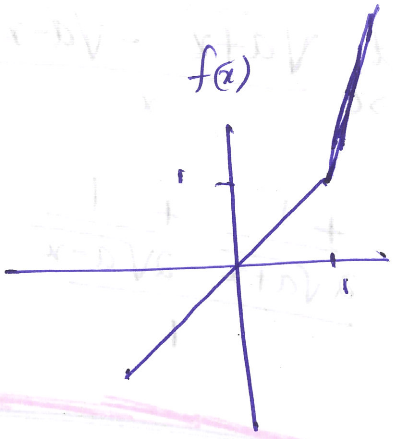
$\therefore (-\infty, 0)$ not disconinuity.

0 discontinuity

$f(g(x)) = \cancel{}$.

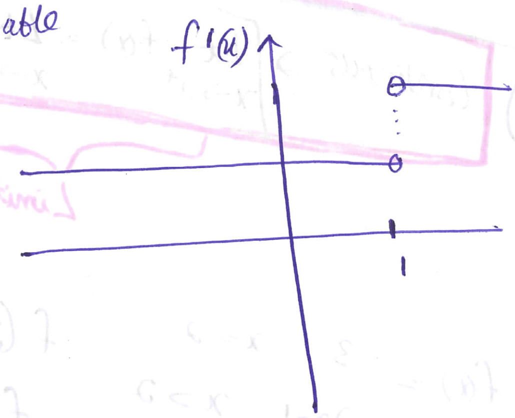
$f(-x) \Rightarrow x^2$

11.) $f(x) = \begin{cases} x & x \le 1 \\ 2x-1 & x > 1 \end{cases}$ @ $x = 1$



$\left.\begin{aligned} f(1^+) &= 1 \\ f(1^-) &= 1 \\ f(1) &= 1 \end{aligned}\right\}$ continous.

$f'(x) = \left.\begin{cases} 1 & x \le 1 \\ 2 & x > 1 \end{cases}\right\}$ not differentiable



12.) $\left.\begin{aligned} f(2^+) &= mx+b \\ f(2^-) &= x^2 \\ f(2) &= x^2 \end{aligned}\right\} \Rightarrow 2m+b = 4$

$f(x) = \begin{cases} x^2, & x \le 2 \\ mx+b, & x > 2 \end{cases}$

$\left.\begin{aligned} f'(x) &= 2x \quad \text{if } x \le 2 \\ f'(x) &= m \quad \text{if } x > 2 \end{aligned}\right\}$ differentiable $\Rightarrow$ smooth curve $\Rightarrow$ slope at 2 is same $\Rightarrow 2x = m$ @ $x = 2$ $\Rightarrow m = 4$

$f'(2^-) = 4$ $\quad \Rightarrow \underline{m=4} \Rightarrow \underline{b = -4}$ option a

$f'(2^+) = m$

13.) a) $f(x) = |x|$ not differentiable at 0

b) $f(x) = \cot x$ not differential at 0

$f'(x) = -\operatorname{cosec}^2 x$

c) $f(x) = \sec x$

$f'(x) = \sec x \tan x$ – differential in $[-1, 1]$

d) $f(x) = \operatorname{cosec} x$ not differentiable at 0

$f'(x) = -\operatorname{cosec} x \cot x$

14) $\sin(x) + 2\sin(2x) + 3\sin(3x) = 8/\pi$

$\Rightarrow \sin x + 2\sin(2x) + 3\sin(3x) - \frac{8}{\pi} = 0$

$\sin(\pi/2) + 2\sin(\pi) + 3\sin\left(\frac{3\pi}{2}\right) - \frac{8}{\pi} = \underset{-8/\pi}{1-3} = \underset{-8/\pi}{-2} < 0$

$\sin(0) + 2\sin(0) + 3\sin(0) - \frac{8}{\pi} = -\frac{8}{\pi} < 0$

$\sin(\pi) + 2\sin(2\pi) + 3\sin(3\pi) = 8/\pi = -8/\pi < 0$

$\therefore$ ~~by elimination~~

$\sin\left(\frac{3\pi}{2}\right) + 2\sin(3\pi) + 3\sin\left(\frac{9\pi}{2}\right) - 8/\pi = -1 + 3 - \frac{8}{\pi}$

$\left(\begin{array}{l} \| \\ 3\sin(4.5\pi) \\ = 3\sin(4\pi + \pi/2) \end{array}\right) \quad = 2 - \frac{8}{\pi} = \frac{2\pi - 8}{\pi} < 0$

$\therefore$ none of the options.

$\sin\left(\frac{\pi}{6}\right) + 2\sin(\pi/3) + 3\sin(\pi/2) - 8/\pi$

$= 3 + (0.5) + \sqrt{3} - \left(\frac{8}{\pi}\right) > 0$

$\therefore$

$f'(x) = \sin x + 2\sin 2x + 3\sin 3x - \left(\frac{8}{\pi}\right)$

$f(x) = \left(-\cos x - \cos 2x - \cos 3x - \frac{8x}{\pi}\right)$

$\left.\begin{array}{l} f(0) = -1-1-1 = -3 \\ f(\pi/2) = 1-4 = -3 \end{array}\right\} \Rightarrow$ atleast one point $f'(c) = 0$

Homogeneous function

the genral form of a homogeneous of degree n is given by:

$f(x,y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots a_{n-1}xy^{n-1} + a_ny^n$

$(a_0 \ldots a_n)$ constants.

$\left.\begin{array}{l} f(x,y) = x^n \phi\left(\frac{y}{x}\right) \\ f(x,y) = y^n \phi\left(\frac{x}{y}\right) \\ f(kx, ky) = k^n f(x,y) \end{array}\right\}$ if $f(x,y)$ is a homogeneous function of degree n.

eg 1) $(x^3 - 2x^2y + 3y^3)$ — 3

eg 2) $\frac{\sqrt{x}+\sqrt{y}}{(x+y)^2} \longrightarrow -3/2$

eg 3) $\sin^{-1}\left(\frac{x^2}{y}\right)$ ✗

eg (4) $e^{x/y} - \log\left(\frac{2x}{y}\right) + 5$ ✗

$$f(x,y) = x^n \phi\left(\frac{y}{x}\right)$$

$$\frac{\partial f}{\partial x} = nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right)$$

$$\frac{\partial f}{\partial y} = x^n \phi'\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right)$$

We always verify that $\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf}$ — ①
if f is a homogenous function of degree n.

$$x\frac{\partial f}{\partial x} = nx^n\phi\left(\frac{y}{x}\right) + x^n\phi'\left(\frac{y}{x}\right)\cdot -\left(\frac{y}{x}\right)$$

$$y\frac{\partial f}{\partial y} = x^n\phi'\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)$$

$$xf_x + yf_y = n\left[x^n\phi\left(\frac{y}{x}\right)\right] = n f(x,y)$$

Similarly We can observe that

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f \text{ — ②}$$

$$\frac{\partial}{\partial x}① \Rightarrow x\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x} + y\frac{\partial^2 f}{\partial x\partial y} = n\frac{\partial f}{\partial x}$$

$$\frac{\partial}{\partial y}① \Rightarrow x\frac{\partial^2 f}{\partial x\partial y} + y\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y} = n\frac{\partial f}{\partial y}$$

$$\Rightarrow x^2\frac{\partial^2 f}{\partial x^2} + x\frac{\partial f}{\partial x} + xy\frac{\partial^2 f}{\partial x\partial y} = nx\frac{\partial f}{\partial x}$$

$$y^2\frac{\partial^2 f}{\partial y^2} + y\frac{\partial f}{\partial y} \quad xy\frac{\partial^2 f}{\partial x\partial y} = ny\frac{\partial f}{\partial x}$$

$$x^2\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2 f}{\partial y^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} = \left[x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial x}\right][n-1]$$

$$\Rightarrow \boxed{x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f}$$

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$U = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then $xU_x + yU_y =$ ______.

Conversion to apply Eulers theorm.

Put $z = \sin u = \frac{x^2+y^2}{x+y}$

$\frac{\partial z}{\partial u} = \cos u$

$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n\left[\frac{x^2+y^2}{(x+y)}\right]$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} \Rightarrow \frac{\partial z}{\partial x}\cdot\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}$

$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} \qquad \frac{\partial z}{\partial y}\cdot\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}$

$\Rightarrow x\cos(u)\frac{\partial u}{u x} + y\cos(u)\frac{\partial u}{\partial y} = n\left[\frac{x^2+y^2}{x+y}\right]$

$\Rightarrow \cos u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \sin u \quad (n=1)$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \underline{\tan u}$

P 23) if $u = x^{-2}\tan(y/x) + 3y^3\sin^{-1}\left(\frac{x}{y}\right)$ | $x>0$ & $y>0$ then $x^2U_{xx} + 2xyU_{xy} + y^2U_{yy} =$ ______.

separately Homgenous $n=-2$

$m=3$

then $u = f(x,y) + g(x,y)$

$\Rightarrow x^2U_{xx} + 2xyU_{xy} + y^2U_{yy} = n(n-1)f(x,y) + m(m-1)g(x,y)$

$\Rightarrow$ Answer $= [-2(-3)]x^{-2}\tan(y/x) + (3\times 2)\cdot 3y^3\sin^{-1}\left(\frac{x}{y}\right)$

$= \underline{6u}$

27) P) if $u = x + \frac{y^2}{x}$, $v = \frac{y^2}{x}$ then $\frac{\partial(u,v)}{\partial(x,y)} = J\left(\frac{u,v}{x,y}\right)$

$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad \begin{vmatrix} 1-\frac{y^2}{x^2} & \frac{2y}{x} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = \underline{\frac{2y}{x}}$

28) $u = 3x+2y-z$, $v = x-y+z$, $w = x+2y-z$

Then $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = J\left(\dfrac{x,y,z}{u,v,w}\right)$

$$= \frac{1}{\dfrac{\partial(u,v,w)}{\partial(x,y,w)}}$$

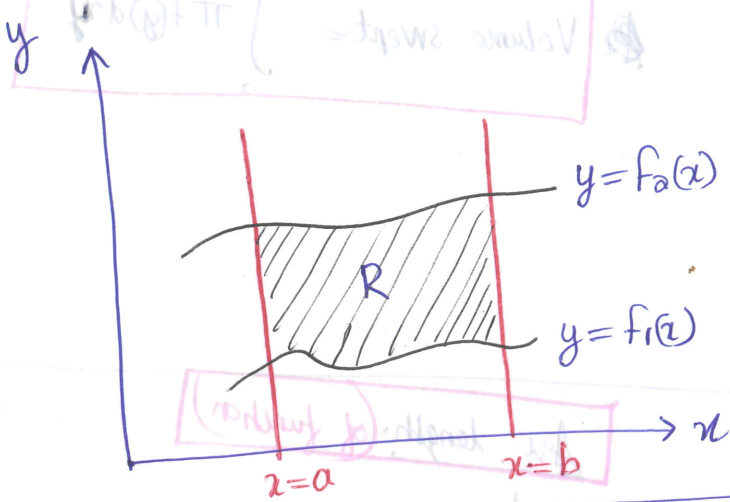
$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= 3(1-2) - 2(-1-1) + (-1)(2+1)$$

$$= -3 \quad +4 \quad -3 = \underline{\underline{-2}}$$

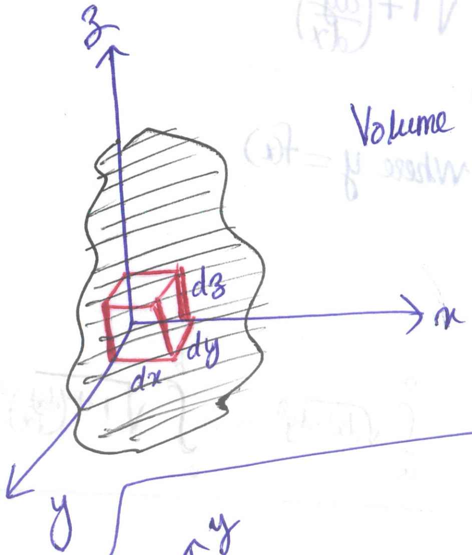
$$\therefore \; J\left(\frac{x,y,z}{u,v,w}\right) = \underline{\underline{\frac{-1}{2}}}$$

Integral Calculus

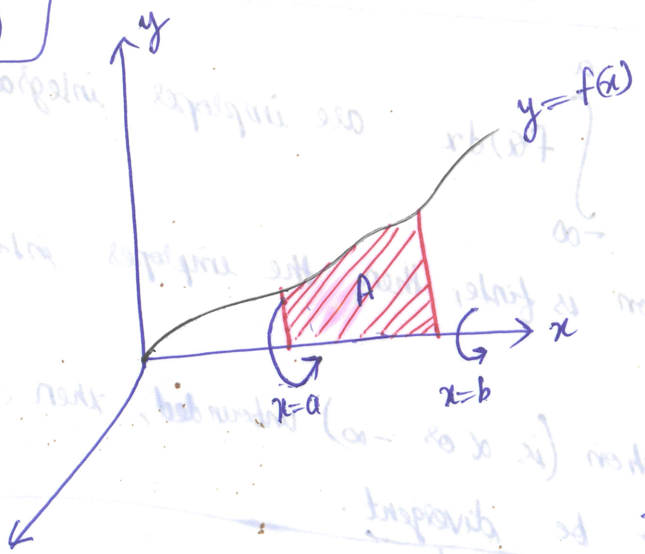


Area of Region R is given by

$$\boxed{\int_{x=a}^{b}\int_{f_1(x)}^{f_2(x)} dy\,dx = \int_a^b \left[f_2(x)-f_1(x)\right]dx}$$



$$\text{Volume } V = \boxed{\int_a^b \left(\int_{f_1(x)}^{f_2(x)} \left(\int_{g_1(x,y)}^{g_2(x,y)} dz\right) dy\right) dx}$$

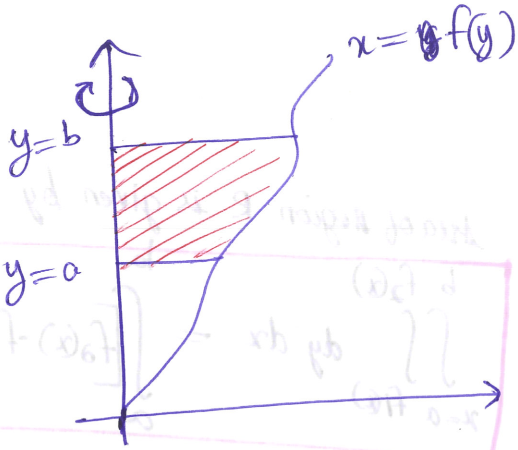


Volume bounded by the rotation of area A around the x axis

$$\int_{x=a}^{x=b}\int_{y=0}^{f(x)}\int_{\theta=0}^{2\pi} r\,dr\,d\theta\,dx$$

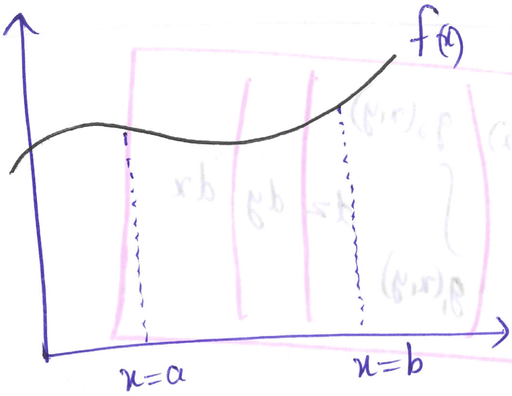
$$= \int_a^b\int_0^{fx} 2\pi \frac{r^2}{2}\,d\theta\,dx$$

$$\boxed{= \int_a^b \pi f^2(x)\,dx}$$



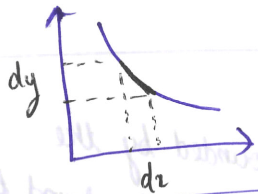
$$\boxed{\text{Volume swept} = \int_a^b \pi f(y)^2 \, dy}$$

Arc length (of function)



$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx$$

where $y = f(x)$



Local linear approximation

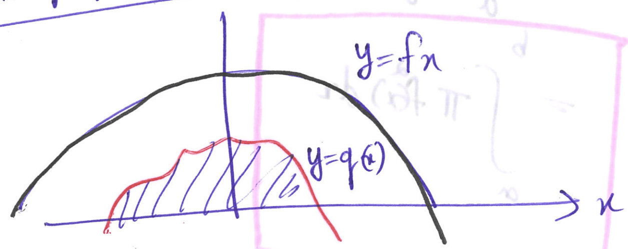
Length dl of arc $= \sqrt{dx^2+dy^2}$

Total length $= \int_{x=a}^{b} dl = \int_a^b \sqrt{dx^2+dy^2} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx$

→ $\int_a^\infty f(x)dx$, $\int_{-\infty}^{\infty} f(x)\,dx$, $\int_{-\infty}^{a} f(x)dx$ are improper integrals.

→ If the final result of integration is finite, then the improper integral is convergent.

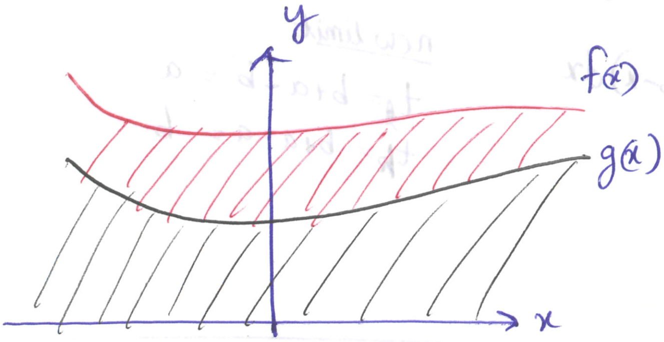
→ If the final result of integration (is ∞ or $-\infty$) unbounded, then the improper integral is said to be divergent.



If $\int_a^b f(x)dx$ is convergent

then $\int_a^b g(x)\,dx$ is also convergent.

provided $\boxed{f(x) \geq g(x)}$



$g(x) \le f(x)$

then if $\int_a^b g(x)\,dx$ is divergent

then $\int_a^b f(x)\,dx$ is divergent.

→ This is called comparison test.

Properties

1) $\int_a^b f(x)dx = -\int_b^a f(x)\,dx$

2) $\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx$

3) $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$

4) $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\,dx$

$= \frac{b-a}{2}$

5) $\int_0^a x\,f(x)dx = \frac{a}{2}\int_0^a f(x)dx$

$\boxed{\text{provided } f(a-x) = f(x)}$

6) $\int_0^{2a} f(x)\,dx =$ → $2\int_0^a f(x)\,dx$ $\quad f(2a-x) = f(x)$

→ 0 $\quad f(2a-x) = -f(x)$

7) $\int_{-a}^{a} f(x) =$ → $2\int_0^a f(x)\,dx$ $\quad f(-x) = f(x)$ Even function! eg. x^2, x^4 etc.

→ 0 $\quad f(-x) = -f(x)$ Odd function! eg x^3, x^5 etc

8) $\int_0^{\pi/2} \sin^m x\,dx = \int_0^{\pi/2} \cos^m x\,dx$

→ $\frac{(m-1)(m-3)(m-5)\cdots 2}{m(m-2)(m-4)\cdots 1}$ 'm' odd

→ $\frac{(m-1)(m-3)\cdots 1}{m(m-2)(m-4)\cdots 2} \times \frac{\pi}{2}$ 'm' even

9) $\int_0^{\pi/2} \sin^m x \cos^n x\,dx$

→ $\frac{(m-1)(m-3)(m-5)\cdots(2 \text{ or } 1)\,(n-1)(n-3)\cdots(1 \text{ or } 2)}{(m+n)(m+n-2)(m+n-4)\cdots(2 \text{ or } 1)}$

→ $\left(\frac{(m-1)(m-3)\cdots(2 \text{ or } 1)\,(n-1)(n-3)\cdots(1 \text{ or } 2)}{(m+n)(m+n-2)(m+n-4)\cdots(1 \text{ or } 2)}\right) \times \pi/2$ [both m & n are even]

2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ put $t = b+a-x$ then $dt = -dx$

$= \int_a^b f(a+b-x)\,dx$

new limit

$t_1 = b+a-b = a$

$t_2 = b+a-a = b$

$$\Rightarrow \boxed{\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx}$$

5) $\int_0^a x f(x)\,dx = \int_0^a (a-x) f(a-x)\,dx = \int_0^a (a-x) f(x)\,dx = a\int_0^a f(x)\,dx - \int_0^a x f(x)\,dx$

ie) $\int_0^a x f(x)\,dx = a\int_0^a f(x)\,dx - \int_0^a x f(x)\,dx \Rightarrow \boxed{\int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx}$

6) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$

$\int_a^{2a} f(x)\,dx = \int_a^{2a} f(2a-x)\,dx$

put $2a-x = u$

$du = -dx$

$dx = -du$

$u_h = 2a-2a = 0$

$u_l = 2a-a = a$

$= \int_a^0 f(u)\,-du = \int_0^a f(u)\,du$

$\Rightarrow \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx$

$$\Rightarrow \boxed{\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx}$$

if $f(2a-x) = f(x)$

if $f(x) = f(2a-x)$

Other wise one more negative comes. Total becomes zero

8) $\int_0^{\pi/2} \sin^m(x)\,dx = \int_0^{\pi/2} \cos^m(x)\,dx$ [because of property 2] $= I$ | Then $2I = \int_0^{\pi/2} [\sin^m(x) + \cos^m(x)]\,dx$

10) $\boxed{\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right]} = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$

Q35) The value of $\int_{-4}^{7} |x|\, dx$ is.

$$= 2\times\int_0^4 x\,dx + \int_4^7 x\,dx = (2\times 8) + \left(\frac{49}{2} - 8\right)$$

$$= 16 + \frac{33}{2} = 16 + 16.5 = \underline{32.5}$$

Q36) Value of $\int_0^{1.5} x[x^2]dx$ $\quad$ $[x]$ is step.

$$= 0\times\int_0^1 x\,dx + \int_1^{\sqrt{2}} x\,dx + 2\times\int_{\sqrt{2}}^{3/2} x\,dx = 0 + \left[\frac{x^2}{2}\right]_1^{\sqrt{2}} + 2\times\left[\frac{x^2}{2}\right]_{\sqrt{2}}^{3/2}$$

$$= 1 - 1/2 + 2\left[\frac{9}{8} - 1\right] = \frac{1}{2} + \frac{1}{4} = \underline{3/4}$$

Q37) $$\int_0^{\pi} x\sin^8(x)\cos^6(x)\,dx = \frac{\pi}{2}\int_0^{\pi} \sin^8(x)\cos^6(x)\,dx = \pi/2\times 2\int_0^{\pi/2} \sin^8(x)\cos^6(x)$$

$$= \frac{\pi}{\cancel{2}}\times\left[\frac{(7\times5\times3\times1)\times(5\times3\times1)}{14\times12\times10\times8\times6\times4\times2}\times\frac{\pi}{2}\right] = \frac{\pi^2}{4}\times\frac{105\times15}{}$$

$$= \underline{\frac{5\pi^2}{4096}}$$

Q40) $$\int_0^{\pi/4} \frac{\sin 2x}{\cos^4 x + \sin^4 x}\,dx = \int_0^{\pi/4} \frac{2\sin x\cos x}{\cos^4 x + \sin^4 x}\,dx = \int_0^{\pi/4} \frac{2\sin x}{\cos^3 x(1 + \tan^4 x)}\,dx$$

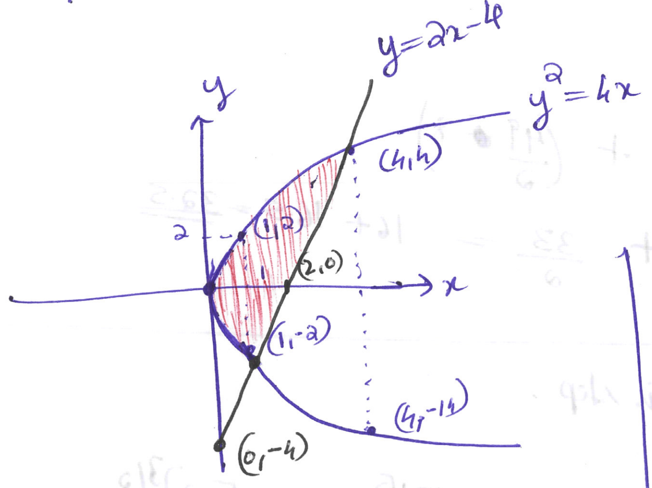
$$\int_0^{\pi/4} \frac{2\tan x\sec^2 x}{1 + [\tan^2 x]^2}\,dx$$

put $\tan^2 x = u$ $\quad$ $\tan^2 0 = u \Rightarrow u_a = 0$

$du = 2\tan x\sec^2 x\,dx$ $\quad$ $\tan^2 \pi/4 = 1 \Rightarrow u_b = 1$

$$= \int_0^1 \frac{du}{1+u^2} = \left[\tan^{-1} u\right]_0^1 = \pi/4 - 0 = \underline{\pi/4}$$

Q41) Area bounded by region bounded by parabola $y^2=4x$ & line $y=2x-4$ is ______.



intersection points

$y^2=4x$

$y^2=4\left(\frac{y+4}{2}\right) \Rightarrow y^2-2y-8=0$

$y=\frac{2\pm\sqrt{4+32}}{2}=\frac{2\pm 6}{2}=4,-2$

$y=4 \to x=4$

$y=-2 \Rightarrow x=1$

points $(1,-2)$ $(4,4)$

$$\text{Area}=\int_{-2}^{4}\int_{x=\frac{y^2}{4}}^{x=\frac{y+4}{2}} dx\,dy$$

$$\text{Area}=\int_{-2}^{4}\left(\frac{y+4}{2}-\frac{y^2}{4}\right)dy=\int_{-2}^{4}\left(\frac{y}{2}+2-\frac{y^2}{4}\right)dy=\left[\frac{y^2}{4}+2y-\frac{y^3}{12}\right]_{-2}^{4}$$

$$=\left(4+8-\frac{16}{3}\right)-\left[1-4+\frac{8}{12}\right]=12-\frac{16}{3}-\left(-3+\frac{2}{3}\right)=\frac{36-16}{3}+\left(\frac{7}{3}\right)$$

$$=\frac{20}{3}+\frac{7}{3}=\frac{27}{3}=9$$

9 units

Q42) The value of the integral $\int_{-\infty}^{0} e^{x+e^x}\,dx$.

$e^{x+e^x}=e^x\cdot e^{e^x}$

$=e^u\,du$

put $e^x=u$

$du=e^x\,dx$

$u_l=e^{-\infty}=0$

$u_h=e^0=1$

$$=\int_0^1 e^u\,du=e^1-e^0=e-1$$

Q43) $\int_{-\infty}^{\infty} \frac{dx}{(1+a^2+x^2)^{3/2}}$ is $= 2\int_0^{\infty} \frac{dx}{(1+a^2+x^2)^{3/2}}$

put $1+a^2=b^2$ $= 2\int_0^{\infty} \frac{dx}{(b^2+x^2)^{3/2}}$

$\Rightarrow 2\int_0^{\pi/2} \frac{1}{b^3} \frac{dx}{(1+\tan^2\theta)^{3/2}}$

$2\int_0^{\pi/2} \frac{1}{b^2}\frac{d\theta}{\sec\theta} = \frac{2}{b^2}\int_0^{\pi/2} \cos\theta \, d\theta$

$= \frac{2}{b^2}\left[\sin\theta\right]_0^{\pi/2} = \frac{2}{b^2} = \frac{2}{1+a^2}$

put $x = b\tan\theta$

$x^2 = b^2\tan^2\theta$

$1+\tan^2\theta = \sec^2\theta$

$b^2(1+\tan^2\theta) = b^2\sec^2\theta$

$\boxed{\begin{matrix}\theta_h = \pi/2 \\ \theta_l = 0\end{matrix}}$ new limit

$dx = b\sec^2\theta \, d\theta$

$\frac{dx}{b\sec^2\theta} = d\theta$

Q44) $\int_0^{\infty} \frac{1}{(x^2+4)(x^2+9)} dx = k\pi$ then $k =$ ______

Solution. $= \frac{1}{5}\int_0^{\infty} \frac{(x^2+9)-(x^2+4)}{(x^2+4)(x^2+9)} = \frac{1}{5}\int_0^{\infty}\frac{1}{(x^2+4)}dx - \frac{1}{5}\int_0^{\infty}\frac{1}{(x^2+9)}dx$

$\frac{1}{5}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^{\infty} = \frac{1}{5}\left[\cancel{\tfrac{1}{2}}(\pi/4) - \cancel{\tfrac{1}{3}}(\pi/6)\right]$

$= \frac{1}{5}\left[\frac{\pi}{4} - \pi/6\right] = \cancel{\frac{10\pi}{8\times18\times5}}\ \frac{2\pi}{8\times18}$ $\quad \frac{2\pi}{24\times5} = \frac{\pi}{60}$

Q 45.) $\int_0^1 x\log x \, dx = \int_0^1 (\log x)\cdot x \, dx$ $= \log x\cdot\frac{x^2}{2} - \int \frac{1}{x}\frac{x^2}{2}$

$= \log x\cdot\left(\frac{x^2}{2}\right) - \int\frac{x}{2}$

$= \left[\log x\left(\frac{x^2}{2}\right) - \frac{x^2}{4}\right]_0^1$

$= 0 - \frac{1}{4}$

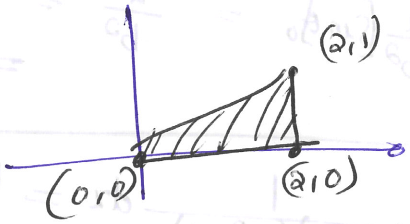
$\boxed{\text{ILATE order}}$

Q50) $\displaystyle\int_{x=1}^{a}\int_{y=1}^{b}\frac{1}{xy}\,dx\,dy = \int_{x=1}^{a}\int_{y=1}^{b}\frac{1}{xy}\,dy\,dx$

$\displaystyle = \int_{x=1}^{a}\frac{1}{x}\left[\log y\right]_1^b\,dx = \int_{x=1}^{a}\frac{1}{x}\log b\,dx = \underline{\log b\cdot\log a}$

Q51) $\displaystyle\int_0^1\int_0^{x^2} x\,dy\,dx \qquad \int_0^1 x\,x^2\,dx = \left[\frac{x^4}{4}\right]_0^1 = \underline{\frac{1}{4}}$

Q52) $\displaystyle\iint_R xy^2\,dR$, where R is Δ with vertices at (0,0), (2,0) & (2,1)



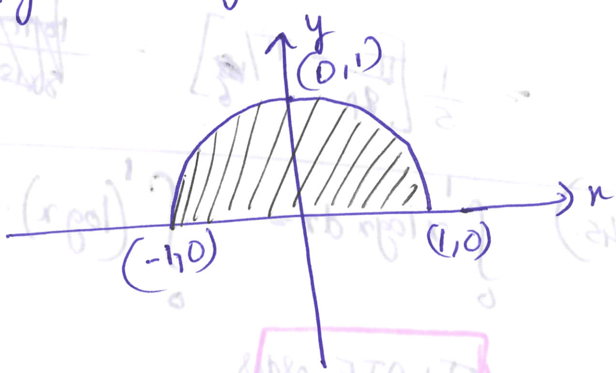
$\displaystyle \text{Area} = \int_{x=0}^{2}\int_{y=0}^{y=x/2}(xy^2)\,dy\,dx$

$\displaystyle \text{Area} = \int_{x=0}^{2}\frac{x^4}{24}\,dx = \left[\frac{x^5}{24\times5}\right]_0^2 = \frac{32}{24\times5} = \underline{\frac{4}{15}}$

$\displaystyle\left[\frac{y^3}{3}\right]_0^{x/2} = \frac{x^3}{24}$

Q53) $\displaystyle\iint_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ R is the region bounded by $x^2+y^2=1\ \forall y\ge 0$

$\displaystyle\int_{x=-1}^{1}\int_{y=0}^{y=\sqrt{1-x^2}}\frac{x^2y^2}{x^2+y^2}\,dy\,dx$



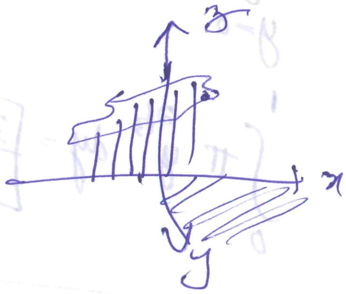
Q55) $\int_0^1\int_0^y\int_0^{1+x+y} f(x,y,z)\,dz\,dx\,dy$ · where $f(x,y,z)=y$ is.

$$\int_0^1\int_0^y y[z]_0^{1+x+y}\,dx\,dy = \int_0^1\int_0^y y\,[1+x+y]\,dx\,dy = \int_0^1\int_0^y (y+yx+y^2)\,dx\,dy$$

$$= \int_0^1 \left[yx+\frac{yx^2}{2}+y^2x\right]_0^y dy = \int_0^1\left(y^2+\frac{y^3}{2}+y^3\right)dy = \left[\frac{y^3}{3}+\frac{y^4}{8}+\frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{3}+\frac{1}{8}+\frac{1}{4} = \frac{8}{24}+\frac{3}{24}+\frac{6}{24} = \underline{\underline{\frac{17}{24}}}$$

Q56) $z = 3+x^2-2y$
Volume under this surface.
over region D defined by $0 \le x \le 1$
$-x \le y \le x$



$$\text{Volume} = \int_{x=0}^{1}\int_{y=-x}^{x}\int_0^{3+x^2-2y} dz\,dy\,dx$$

$$= \int_0^1\int_{-x}^{x}(3+x^2-2y)\,dy\,dx = \int_0^1\left[3y+x^2y-y^2\right]_{-x}^{x}dx$$

$$= \int_0^1\left[6x+x^2(x+x)-\cancel{x^2+x^2}\right]dx = \int_0^1(6x+2x^3)\,dx$$

$$= \left[3x^2+\frac{x^4}{2}\right]_0^1 = 3+\frac{1}{2} = \underline{\underline{7/2}}$$

Q57) $\int_{x=0}^{3}\sqrt{1+(y')^2}\,dx$

$y = \frac{2}{3}x^{3/2}$

$y' = x^{1/2}$

$$\int_{x=0}^{3}\sqrt{1+x}\ d\,dx = \frac{2}{3}\left[(1+x)^{3/2}\right]_0^3 \qquad \frac{2}{3}\left(4^{3/2}-1\right)$$

$$= \frac{2}{3}(8-1) = \frac{7\times 2}{3} = \underline{\underline{\frac{14}{3}}}$$

58) $y = \sqrt{x} \quad 0 < x < 4$

$y^2 = x$

$$= \int_0^4 \pi x \, dx = \left[16\right]\frac{\pi}{2} = \underline{\underline{8\pi}}$$

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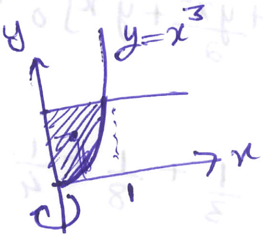
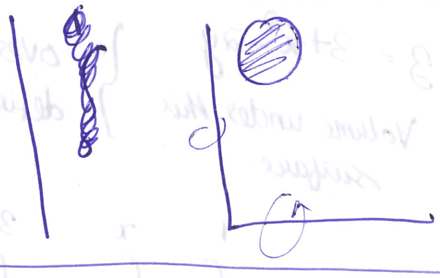
59) $y = x^3$ between y axis and $y = 1$ about y axis.



$$\int_{y=0}^{1} \pi(\sqrt[3]{y})^2 \, dy \qquad x = \sqrt[3]{y}$$

$$\int_0^1 \pi y^{2/3} \, dy = \left[\frac{3\pi}{5} y^{5/3}\right]_0^1 = \frac{3\pi}{5}$$



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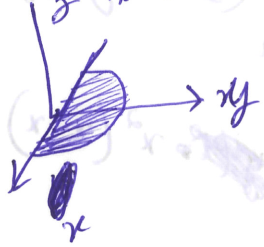
53) Value of integral $\iint_R \frac{x^2y^2}{x^2+y^2} \, dx\, dy$; R is bounded by $x^2+y^2=1$, $y \geq 0$.



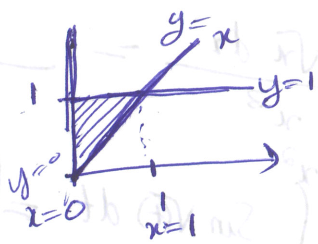
$$\int_{r=0}^{1}\int_{\theta=0}^{\pi} \frac{r^2\cos^2\theta \cdot r^2\sin^2\theta}{\cancel{r^2}} \, r\, dr\, d\theta$$

$$\int_{r=0}^{1}\int_{\theta=0}^{\pi} r^3 \sin^2\theta \cos^2\theta \, d\theta\, dr$$

$$= \int_{r=0}^{1} r^3 \left[\int_0^{\pi} \sin^2\theta \cos^2\theta \, d\theta\right] dr = 2 \times \left[\frac{r^4}{4}\right]_{r=0}^{1} \int_0^{\pi/2} \sin^2\theta \cdot \cos^2\theta \, d\theta$$

$$= \frac{1}{2} \cdot \frac{\pi}{2}\left[\frac{1}{2\times 4}\right] = \underline{\underline{\pi/32}}$$

54) $\int_0^1 \int_x^1 e^{y^2}\, dy\, dx$



$\frac{J(y,x)}{J(x,y)} = -1$

$= \int_{y=0}^{1} \int_{x=0}^{x=y} e^{y^2}\, dx \cdot dy$

$= \int_0^1 \left[[x]_0^y e^{y^2} \right] dy = \int_0^1 y e^{y^2} \cdot dy$

put $y^2 = u$

$2y\,dy = du$

$y\,dy = \frac{du}{2}$

$u_l = \sqrt{y} = 0$

$u_h = \sqrt{y} = 1$

$= \frac{1}{2} \int_0^1 e^u du = \left[\frac{e^u}{2}\right]_0^1 = \left[\frac{e^{y^2}}{2}\right]_0^1 = \frac{e-1}{2}$

~~55)~~ 49) $fa = \int_0^\infty e^{-x} \frac{\sin ax}{x} dx$ then $f(a) =$ ______

$\frac{\partial f}{\partial a} = \int_0^\infty \frac{e^{-x} \cancel{x} \cos ax}{\cancel{x}} dx = \int_0^\infty e^{-x} \cos ax\, dx$

$\frac{\partial f}{\partial a} = \left[\frac{e^{-x}}{1+a^2}(-1 \cos ax + a \sin ax)\right]_0^\infty = \frac{1}{1+a^2}$

$\frac{\partial f}{\partial a} = \frac{1}{1+a^2}$

$f(a) = \int \frac{1}{1+a^2} da = \tan^{-1} a + c$

38) $x \sin(\pi x) = \int_0^{x^2} f(t)\, dt$ (Property 10)

39) $\lim_{x \to 0} \int_0^{x^2} \frac{\sin\sqrt{x}\, dx}{x^3} =$ ______

Solution

$$\frac{d}{dx}\int_0^{x^2} \sin\sqrt{t}\, dt = \left(2x\sin\sqrt{x^2}\right) - \sin 0$$

$$\lim_{x\to 0} = \frac{2x\sin x}{3x^2} = \frac{2\sin x}{3x} = \frac{2}{3}$$

---

46) The integral $\int_1^3 \frac{\sqrt{1+x}}{(x-1)^2}\, dx$

Comparison

$$\int \frac{1}{(x-1)^2}\, dx \leq \int \frac{\sqrt{1+x}}{(x-1)^2}\, dx \quad \forall x \in (3,1)$$

$$\infty \qquad\qquad \infty$$

$$\int_1^3 \frac{1}{(x-1)^2}\, dx < \int_1^3 \frac{\sqrt{1+x}}{(x-1)^2}\, dx$$

$$= \left[\frac{-1}{(x-1)}\right]_1^3 = -\left[\frac{1}{2} - \infty\right] = \infty \quad \therefore \text{ always divergent.}$$

---

47) $f(a) = \int_1^2 \frac{x^3+1}{\sqrt{2-x}}\, dx > \int_1^2 (x^3+1)\, dx$

$\left[\frac{x^4}{4} + x\right]_1^2 dx$

$= (16+2) - \left(\frac{1}{4}+1\right)$

$= 18 - 1.25$ finite.

$\boxed{\begin{array}{l} 0.5 > 0.1 \\ \frac{1}{0.5} < 0.1 \\ 2 < 0.1 \end{array}}$

$$\int_1^2 \frac{1}{\sqrt{2-x}}\, dx < f(a)$$

$$\left[-2\sqrt{2-x}\right]_1^2 = +2 \quad \text{no conclusion.}$$

48) The integral $\int_{1}^{\infty} \frac{e^{-x}}{x^2} dx$

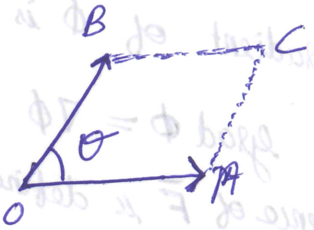
Vector calculus.

- Then a unit vector in the direction of $\overline{OP}$ = given by $\frac{\overline{OP}}{|\overline{OP}|} = \frac{\bar r}{r}$

$r = \sqrt{x^2+y^2+z^2}$

- Let $\overline{OA} = \bar a = a_1\hat i + a_2\hat j + a_3\hat k$

  $\overline{OB} = \bar b = b_1\hat i + b_2\hat j + b_3\hat k$



With an angle between them as θ as shown in figure.

$$\boxed{\underset{n\to\infty}{Lt}\ \sqrt[n]{f(b)}}$$

- The scalar product of $\bar a$ & $\bar b$ is denoted by $\bar a\cdot\bar b$

$\bar a\cdot\bar b = ab\cos\theta = a_1b_1 + a_2b_2 + a_3b_3 = \bar b\cdot\bar a$

- Similarly the vector product of $\bar a$ & $\bar b$ is denoted and is defined by

$$\hat a\times\hat b = ab\sin\theta\,\hat n = \begin{vmatrix}\hat i & \hat j & \hat k\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Where $\hat n$ is a unit vector, which is $\perp$r to both $\bar a$ & $\bar b$ and is given by $\hat n = \frac{\bar a\times\bar b}{|\bar a\times\bar b|}$

$|\bar a\times\bar b| = ab\sin\theta$ = area of parallelogram OABC

Let $\phi(x,y,z) =$ be a ~~[struck text]~~ scalar function

$$\bar{F} = F_1(x,y,z)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\hat{k}$$

be a defined at all points inside C;

the gradient of ϕ is defined as

$$\text{grad}\,\phi = \nabla\phi$$

Divergence of $\bar{F}$ is defined as

$$\text{Div}\,\bar{F} = \nabla\cdot\bar{F}$$

Curl of $\bar{F}$ is denoted by

$$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$V = \phi(x,y,z)$

$\boxed{(\nabla(\phi))\cdot \hat{a_n}}$ directional derivative of ϕ in $\hat{a_n}$ direction

$\hat{a_n}\cdot(\nabla\phi)$

$\nabla\phi$ is a vector which is always in the direction of normal vector and $\frac{\nabla\phi}{|\nabla\phi|}$ is called unit normal vector to the surface $\phi(x,y,z) = c$

$\nabla\phi\cdot\hat{a}$ is directional derivative of ϕ in the direction of an unit vector $\hat{a}$. The maximum value of the directional derivative of ϕ is given by $|\nabla\phi|$ and occurs always in the the direction of $\nabla\phi$.

$$\boxed{\nabla\cdot A = \underset{\Delta V\to 0}{Lt} \frac{\oint D\cdot ds}{\Delta V}}$$

- Suppose divergence of $\bar{F} = 0$ ie $\nabla \cdot F = 0$ then we say that $\bar{F}$ is incompressible vector or solenoidal vector
- curl $\bar{F} = \bar{0} \Rightarrow \nabla \times \bar{F} = 0$, then $\bar{F}$ is said to be irrotational.

~~We always observe that~~ divergence of $\bar{F} =$ ~~$\nabla \times$~~ $\nabla \cdot (\nabla \times \bar{F}) = 0$

- If $\bar{F}$ is solenoidal then $\nabla \cdot \bar{F} = 0$

$$\Rightarrow \nabla \cdot (\nabla \times \bar{A}) = 0 \Big\} F \text{ is solenoidal.}$$

where $F = \nabla \times \bar{A}$

$A = A_i \hat{i} + A_j \hat{j} + A_z \hat{z}$

$$\nabla \times \bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_i & A_j & A_z \end{vmatrix} = \begin{array}{l} \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_j}{\partial z}\right) = \hat{i} P_x \\ -\hat{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_i}{\partial z}\right) = \hat{j} P_y \\ \hat{k}\left(\frac{\partial A_j}{\partial x} - \frac{\partial A_i}{\partial y}\right) = \hat{k} P_z \end{array}$$

$$\nabla \cdot (\nabla \times A) = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}$$

$$= \frac{\partial^2 A_z}{\partial x \partial y} - \boxed{\frac{\partial^2 A_j}{\partial x \partial z}} - \frac{\partial^2 A_z}{\partial y \partial x} + \boxed{\frac{\partial^2 A_i}{\partial y \partial z}} + \boxed{\frac{\partial^2 A_j}{\partial z \partial x}} - \boxed{\frac{\partial^2 A_i}{\partial z \partial y}}$$

$$\nabla \cdot (\nabla \times A) = 0$$

→ Similarly $\nabla \times (\nabla \phi) = 0$

where $\nabla \phi =$ and irrotational vector.

→ $\nabla \cdot (\nabla \times A) = 0$ where $\nabla \times A =$ sourceless vector.

$$\boxed{\nabla \cdot (\nabla \phi) = \text{laplacian} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}}$$

divergence of gradient always a scalar

$$\boxed{\nabla(\nabla \cdot A) = \nabla \times \nabla \times A - \nabla^2 A}$$

gradient of divergent always a vector.

where $\nabla^2 A$ is vector laplacian of A

given by $\boxed{\nabla \cdot (\nabla A_x)\hat{i} + \nabla \cdot (\nabla A_y)\hat{j} + \nabla \cdot (\nabla A_z)\hat{z}}$

ie, **laplacian of vector = scalar laplacian of each component in the respective direction.**

if $\nabla^2\phi = 0 \Rightarrow \nabla\cdot(\nabla\phi) = 0$

$\boxed{\text{then we say } \phi \text{ is satisfying laplace equation. } \phi \text{ is a harmonic function. Hence grad } \phi - \nabla\phi \text{ is both solinoidal \& irrotational.}}$

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Q 60, 72

$\phi = x^2 + y^2 + z^2 = 9$

Q60) $\phi(x,y,z) = c$

unit normal given by $\frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4(x^2+y^2+z^2)}}$

$$\frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2\sqrt{9}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{3}$$

Q61) $\nabla\phi = (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}\Big|_{1,-2,-1} = (4+4)\hat{i} - \hat{j} + (-10)\hat{k} = 8\hat{i} - \hat{j} - 10\hat{k}$

$(\nabla\phi)\cdot\hat{n}$ = directional derivative.

$$\hat{n} = \frac{2i - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{3}$$

$$(\nabla\phi)\cdot\hat{n} = \frac{16+1+20}{3} = \frac{37}{3}$$ option b

---

Q62) $\nabla\phi = \left(ye^{x}(\sin(x+y)) + e^{xy}\cos(x+y)\right)\hat{i} + xe^{y}\sin(x+y) + e^{xy}\cos(x+y)\hat{j}$

$$\nabla\phi\Big|_{0,\pi/2} = \left(\frac{\pi}{2}\cdot e^0\left[\sin\left(\frac{\pi}{2}\right)\right] + e^0\cos(\pi/2)\right]i + \left[0e^{\pi/2}\sin(\pi/2) + e^0\cos(\pi/2)\right]j$$

$$\nabla\phi\Big|_{0,\pi/2} = \frac{\pi}{2}\hat{i} + 0 = \frac{\pi}{2}i$$

Q63) $\nabla \times \bar{V}$ $\Rightarrow$ $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

$A_x = (x+6y+az)$

$A_y = (2bx-3y+2z)$

$A_z = 4x+2cy+z$

$\nabla \times \bar{V} = \hat{i}(2c-2) - \hat{j}(4-a) + \hat{k}(2b-6)$

$\Rightarrow \underline{c=1 \quad a=4 \quad b=3}$ option b

Q64) $\bar{V} = e^x \hat{i} + 2y\hat{j} - \hat{k}$ is

$\nabla \cdot \bar{V} = e^x + 2 \neq 0$

$\nabla \times \bar{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = i(0-0) - j(0-0) + \hat{k}(0-0)$

$= 0$

$\therefore$ Irrotational but not divergence free option b

Q65.) If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ & $r = |\bar{r}|$, then $\Delta(\sin r) = ?$

$\boxed{\nabla \phi = \frac{1}{h_1}\phi_1' \hat{u}_1 + \frac{1}{h_2}\phi_2' \hat{u}_2 + \frac{1}{h_3}\phi_3' \hat{u}_3}$

$|\bar{r}| = \sqrt{x^2+y^2+z^2}$

$\sin r = \sin\sqrt{x^2+y^2+z^2} = \sin r$

$\Rightarrow \Delta(\sin r) = \cos r \; \hat{r} = \frac{\cos r \; \bar{r}}{r}$

Q66) If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\bar{r}|$ $\operatorname{div}(e^r \bar{r}) =$

$e^r \bar{r} = e^{\sqrt{x^2+y^2+z^2}}\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$

$\nabla \cdot (e^r \bar{r}) = e^{\sqrt{x^2+y^2+z^2}} \cdot \left[\left(\frac{1(2x^2)}{2\sqrt{x^2+y^2+z^2}} + 1\right) + \left(\frac{1\, 2y^2}{2r} + 1\right) + \frac{2z^2}{2r} + 1\right]$

$= e^r\left[\frac{2r^2}{2r} + 3\right] = \underline{(3+r)e^r}$

67.) if $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ & $r = |\bar{r}|$ then

$\nabla \times (r^4 \bar{r})$

$\nabla \times \left[(x^2+y^2+z^2)^2 \bar{r} \right]$

$= x^2+y^2+z^2 \left[\nabla \times \bar{r} \right]$

$= \underline{\underline{0}}$

$$\nabla \times r = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$= i(0) + j(0) + \hat{k}(0)$

68.) If $2z\hat{i} + 2y\hat{j} + 2x\hat{k}$

$\nabla (A_r, \phi, z) = \frac{1}{r} \frac{\partial (A_r)}{\partial r} \cdot \hat{r}$

$\nabla \cdot (A) = \frac{1}{h_2 h_3} \frac{\partial (h_1 A_1)}{\partial u_1} \cancel{\hat{e}_1}$

$\frac{1}{h_1 h_3} \frac{\partial}{\partial u_2} (h_2 A_2) \cancel{\hat{e}_2}$

$\frac{1}{h_2 h_1} \frac{\partial}{\partial u_3} (h_3 A_{u_3}) \cancel{\hat{e}_3}$

$$\boxed{\frac{r, \phi}{r^2 \sin\phi}}$$

$\nabla \phi = \frac{1}{h_1} \frac{\partial \phi}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial \phi}{\partial u_3} \hat{e}_3$

68) $\bar{F} = 2z\hat{i} + 2y\hat{j} + 2x\hat{k}$ $\quad \oint_C F\cdot d\bar{s}$ along

c from $(0,2,1)$ to $(4,1,-1)$

$$\bar{ds} = \frac{4\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{4\hat{i} - \hat{j} - 2\hat{k}}{3} \quad X$$

Or $\displaystyle\int_C \bar{F}\cdot ds = 2\int z\,dx + y\,dy + x\,dz = \int y\,dy + d(xz)$

$$= 2\left[\frac{y^2}{2} + xz\right]_{(0,2,1)}^{4,1,-1} \rightarrow = 2\left[\frac{1}{2} + (-4) - 2\right] = -1 - 12 = \underline{-11}$$

69) $\displaystyle\int_A^B \bar{F}\cdot dx = \int_A^B F_1 dx + F_2 dy + F_3 dz$

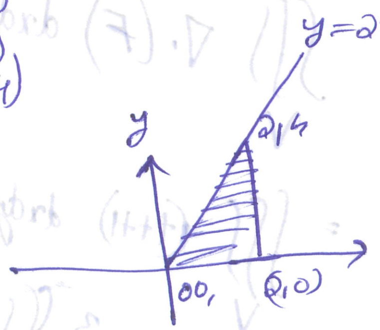
$$= \int_A^B (2xy + z^3)\,dx + \cancel{3x}\, x^2\,dy + 3xz^2\,dz = \int_A^B d(xz^3) + d(x^2y)$$

$$= \left[xz^3 + x^2y\right]_{1,-2,1}^{3,1,4} = (3\times 64) + (9\times 1) - (1 + -2)$$

$$= 192 + 9 - 1 + 2$$

$$= \underline{\underline{202}}$$

70 $\displaystyle\int_R (x+y)\,dx + x^2\,dy$ $\quad$ R is triangle $(0,0)$ $(2,0)$ $(2,4)$



Green's theorem

$$= \iint_S \cancel{(2x+y)}\,(2x-1)\,dx\,dy$$

$$= \int_0^2 \int_0^{2x} (2x-1)\,dy\,dx$$

$$\int_0^2 (2x-1)[y]_0^{2x}\, dx = \int_0^2 (2x-1)\cdot 2x\, dx$$

$$= \int_0^2 4x^2 - 2x\, dx = \left[\frac{4x^3}{3} - \cancel{2}x^2\right]_0^2 = \frac{32}{3} - 4$$

$$= \cancel{0}\ \frac{20}{3} = 6.666$$

71) $\int [(2x-y)dx + (x+3y)dy]$ over ellipse $x^2+4y^2=4$

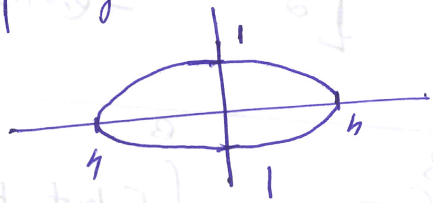
$$\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$$

$$= \iint_R 2\, dx\, dy \quad \text{green's theorem}$$

$y=0$
$x=4$
$x=0$
$y=1$

$$= 2\iint dx\, dy = 2[\text{Area of ellipse}]$$

$$= 2\pi ab = 2\pi \times 2 = 4\pi$$



72) $\bar{A} = \nabla\phi$ then $\int_C \bar{A}\cdot d\bar{s}$ where $C = \frac{x^2}{4} + \frac{y^2}{9} = 1$

$$= \iint_S (\nabla\times\bar{A})\cdot ds = \iint_S \nabla\times(\nabla\phi)\cdot ds = 0$$

73) $\iint_S \bar{F}\cdot\bar{N}(ds)$ $\quad S =$ surface of sphere $x^2+y^2+z^2=4^2$

$$\bar{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\iiint_V \nabla\cdot(F)\, dx\,dy\,dz$$

$$= \iiint_V (1+1+1)\, dx\,dy\,dz$$

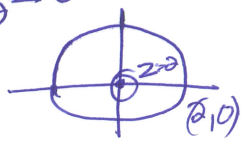
$$= 3\iiint_V dx\,dy\,dz = 3\times\frac{4\pi r^3}{3} = 4\pi r^3 = 64\times 6\times\pi = 256\pi$$

Assume $\bar{N}$ is the $\hat{n}$ unit vector of ds.

75) $\bar{F} = (x^2+yz)\hat{i} + (y^2+xz)\hat{j} + (z^2+xy)\hat{k}$ be the differentiable vector point function.

$\oint f \cdot ds$ where C is curve $x^2+y^2=4$, $z=2$ is ____.

$dz = 0$



Solution $\oint (x^2+yz)dx + (y^2+xz)dy + (z^2+xy)dz$

$= \oint (x^2+yz)dx + (y^2+xz)dy$

$= \iint_S (z-z)\, dx\,dy = 0.$

Q74)

Fourier Series.

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega x) + b_n \sin(n\omega x)\right]$$

$$\omega = \frac{2\pi}{T}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right)\right]$$

$T \rightarrow$ fundamental period.

$$a_0 = \frac{1}{T}\int_0^T f(x)\,dx$$ DC component.

Limit change

$x = T$

$\omega x = \omega T = 2\pi$

$$b_n = \frac{2}{T}\int_0^T f(x) \sin(n\omega x)\,dx$$

put $\omega x = u$

$dx = \frac{du}{\omega} = \frac{du}{\frac{2\pi}{T}}$

$$b_n = \frac{2}{\cancel{T} \times \frac{2\pi}{\cancel{T}}}\int_0^{2\pi} f(x) \sin(n\omega x)\,d(\omega x)$$

$$b_n = \frac{2}{2\pi}\int_0^{2\pi} f(x) \sin(n\omega x)\,d(\omega x)$$

$$b_n = \frac{2}{2\pi}\int_0^{2\pi} f(\omega x) \sin(n\omega x)\,d(\omega x)$$

$$a_n = \frac{2}{T}\int_0^T f(x) \cos(n\omega x)\,dx$$

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} f(x) \cos(n\omega x)\,d(\omega x)$$

---

if $f(x)$ is even

$$\int_0^T f(x) \sin(n\omega x)\,dx = 0$$

$\boxed{\text{even} \times \text{odd} = \text{odd}}$

$$\therefore f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega x)\right]$$ Half Range even

if $f(x)$ is odd

$$\int_0^T f(x) \cos(n\omega x)\,dx = 0$$

odd $\times$ even = odd

odd $\times$ odd = even

$$\Rightarrow f(x) = \sum_{n=1}^{\infty}\left[b_n \sin(n\omega x)\right]$$ Half Range odd.

77) Coefficient of $\sin 2x$ in Fourier of $f(x) = x^2\cos(x)$ in $(-\pi, \pi)$ is.

$\downarrow$ evenfunction

$\therefore$ No sine terms $= 0$

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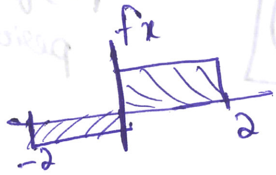
78) The term independent of x in Fourier series of $f(x) = \begin{cases} -1 & -2 \le x \le 0 \\ 2 & 0 < x \le 2 \end{cases}$



DC component $= \dfrac{2\times 2 - 2\times 1}{4} = \dfrac{4-2}{4} = 0.5$

$\therefore$ $a_0 = 1/2$

---

79) $f(x) = \begin{cases} -3, & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases}$ $\rightarrow$ Average / DC component 0

$\downarrow$

odd $\therefore$ only cosine terms & NO DC

option b

---

80) $f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$ is given by

$\downarrow$ odd

No DC component

$$f(x) = \sum_{n=1}^{\infty} \frac{k}{\pi}\left[\frac{2-2(-1)^n}{n}\right]\sin nx$$

$$f(\pi/2) = \sum_{n=1}^{\infty} \frac{k \cdot 2}{\pi}\left[\frac{1-1(-1)^n}{n}\right]\sin(n\pi/2)$$

given series $g(x) = \sum_{n=1}^{\infty} \frac{1}{2}\left[\frac{1-1(-1)^n}{n}\right]\sin(n\pi/2)$

$$\therefore \frac{f(x)}{g(x)} = \frac{k\cdot 4}{\pi}$$

$$\frac{f(\pi/2)\pi}{k\cdot 4} = g(x) = \frac{k\cdot\pi}{k\cdot 4} = \pi/4$$

81) $b_1 = \frac{2}{2\pi}\int_0^{2\pi} f(x)\cdot \sin(\omega x)\, d\omega x = \frac{4}{2\pi}\int_0^{\pi} f(x)\cdot \sin(\omega x)\, d\omega x$

$= \frac{4}{2\pi}\int_0^{\pi} x(\pi - x)\sin(\omega x)\, d\omega x = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin$

put $\omega x = u$, $x = \frac{u}{\omega}$ $\quad = \frac{4}{2\pi}\int_0^{\pi}\frac{u}{2}\left(\pi - \frac{u}{2}\right)\sin u\, du$

$\boxed{\text{here } \omega = \frac{2\pi}{T} = \frac{2\pi}{\ } \quad \omega = \frac{2\pi}{\pi} = 2}$

$=$

Integral calculus

$\rightarrow$ $f(x)$ is continous on $[a,b]$
$F(x)$ is the antiderivative of $f(x)$ on $[a,b]$
ie $\int f(x)dx = F(x)$

then $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$

Newton - Leibnitz formula

Theorm:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = \frac{d}{dx}\left[F[v(x)] - F[u(x)]\right] = F'\Big|_{v(x)}\frac{dv}{dx} - F'\Big|_{u(x)}\frac{du}{dx}$$

$$\Rightarrow \boxed{\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}}$$

$$\Rightarrow \boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

Mean value theorm for integrals

$F(x)$ is continous in $[a,b]$
$F'(x) = f(x)$ is continous in (a,b)

$$\Rightarrow f(c) = \frac{F(b) - F(a)}{b-a}$$

atleast one c $(\in a,b)$ satisfies this

$$\Rightarrow (b-a) f(c) = F(b) - F(a)$$

from Newton-Leibnitz formula

$$\boxed{(b-a) f(c) = \int_a^b f(x)\,dx}$$

atleast one $c \in (a,b)$ satisfies this.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$\Rightarrow$ $\boxed{f(c) = \text{Average of mean of } f(x) \text{ in } (a,b)}$

Leibnitz Rule.

$f(x,y)$ & $\frac{\partial f}{\partial y}$ be continous functions of x,y then

$\boxed{f_y(x,y) = \frac{\partial f(x,y)}{\partial y}}$

$$\boxed{\frac{d}{dy}\left[\int_a^b f(x,y)\,dx\right] = \int_a^b f_y(x,y)\,dx}$$

$\boxed{\text{* } a,b \text{ constants only}}$.

* Order of differentiation & integration can be interchanged if they are operating on different variable for multi variable function

Properties of Definite Integrals

1.) $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$ Change of Variables.

2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ Limit Exchange property.

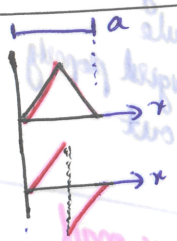
3) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad (a<c<b)$ Divide & Rule property

4) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ Reverse Cowgirl property.

5) $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\,dx = \frac{b-a}{2}$ Averaging out property.

6) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & f(x) = \text{even},\ f(-x) = f(x) \\ 0 & f(x) = \text{odd},\ f(-x) = -f(x) \end{cases}$ Odd even property

7.) $\int_0^a f(x)\,dx = \begin{cases} 2\int_0^{a/2} f(x)\,dx & f(a-x) = f(x) \\ 0 & f(a-x) = -f(x) \end{cases}$



Triangle-hourglass property

8.) $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$ if $f(x+a) = f(x)$ Periodic function property.

9.) $\int_a^b x f(x)\,dx = \frac{a+b}{2}\int_a^b f(x)\,dx$ if $f(a+b-x) = f(x)$

X-triangle function property.

10) $$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\frac{2}{3} & \text{if } n \text{ is odd.} \\ \left[\left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\frac{1}{2}\right]\times\frac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

$(\sin/\cos)^n$ property

$\sin$ or $\cos$ en property.

11. $$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{(m-1)(m-3)(m-5)\ldots(2\text{ or }1)\,(n-1)(n-3)(n-5)\ldots(2\text{ or }1)}{(m+n)(m+n-2)(m+n-4)\ldots\ldots(2\text{ or }1)}\,K$$

$K = \pi/2$ if m & n even

$K = 1$ otherwise.

Sin (em a) Cos (en a) property!

Trick to remember

1) Properties due to definition
1) Change of Variable
2) Limit exchange
3) Divide & Rule
4) Reverse Longird property
5) Averaging out

2) Properties due to function
1) Odd even property
2) Triangle hourglass property
3) X triangle function property
4) Periodic function property

Properties of Sinusoidal
1) Sin or Cos en
2) Sin (em a) Cos (en a)

Anonymous mail

$$\boxed{\int_0^{\infty} e^{-x} x^n\,dx = n!}$$

Improper Integrals & their Convergence

First kind of Improper Integrals

$\int_a^b f(x)dx$ = Improper

if either $a =$ (OR) $b = \pm\infty$ or both

Second kind of Improper Integrals

$\int_a^b f(x)dx$ is said to be improper

- $f(x) = \pm\infty$ for some $x \in [a,b]$

* closed interval note means including a & b if some $f(x)$ is not limited.

Tests for Convergence

Comparison test

I) if $0 \le f(x) \le g(x)$ $\forall x \in [a,b]$ &

$\int_a^b g(x)dx$ converges $\Rightarrow \int_a^b f(x)dx$ converges.

Statement

* If a function $g(x)$ which is alway greater than given function $f(x)$ $\forall$ values of x in an closed interval $[a,b]$ then

$\int_a^b g(x)dx$ converges $\Rightarrow \int_a^b f(x)dx$ converges

II) if $0 \le g(x) \le f(x)$ $\forall x \in [a,b]$ &

$\int_a^b g(x)dx$ diverges $\Rightarrow \int_a^b f(x)dx$ diverges.

If a function $g(x)$ which is always less than given function $f(x)$ $\forall$ values of x in a closed interval $[a,b]$ then $\int_a^b g(x)dx$ diverges

$\Rightarrow \int_a^b f(x)dx$ diverges

Limit Comparison test

Let $f(x)$ & $g(x)$ be two positive functions and

i) $f(x) \to \infty$ as $x \to a$ but $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = l$ (FINITE & NON-ZERO)

OR ii) $f(x) \to \infty$ as $x \to b$ but $\lim\limits_{x\to b} \frac{f(x)}{g(x)} = l$ (FINITE & NON-ZERO)

b can be infinite also.

$\Rightarrow$ then $\int_a^b f(x)dx$ & $\int_a^b g(x)dx$ $\boxed{\text{both converge or diverge together.}}$

finite & non zero because they are interchangable and result is also valid.

Comparison Results

1.) $\int_a^\infty \frac{dx}{x^p}$ $(a>0)$ converges to $\frac{1}{p-1\,a^{p-1}}$ $\forall (p>1)$

diverges for $p \le 1$

2) $\int_a^\infty e^{-px}dx$ & $\int_{-\infty}^b e^{px}dx$ converges for $p>0$ & diverges for $p<0$

3. $\int_a^b \frac{dx}{(b-x)^p}$ Converges to $\frac{1}{(1-p)(b-a)^{p-1}}$ when $p<1$

diverges to ∞ when $p \geq 1$

4) $\int_a^b \frac{dx}{(x-a)^p}$ Converges to $\frac{1}{(1-p)(b-a)^{p-1}}$ When $p<1$ &

diverges to ∞ when $p \geq 1$

Procedure to find Convergence

Step 1 : Examine the improper integral and see if its possible to find the result of integration

Step 2 : If the given function is discontinous at any point within the limit then split the integral using "divide and rule" properly and then each part must be examined.

* Splitting must be such that the discontinous point is at the Limit of integral

Step 3 : If result of integration cannot be found they compare with the above four standard form/or any function whose convergence may be known. see example — 47, 48, 49, 50, 51, 52, 53, 54 Text book.

Multiple Integral

Double integral / triple integral.

→ A multiple integral is first evaluated with respect to the variable whose limits are a function of remaining variables that has not been integrated yet and last with respect to variable which has constant limits.

→ If all variable have constant limits then order of integration does not matter.

eg: $$\iiint_V f(x,y,z)\,dV = \int_{x=c_1}^{c_2}\left[\int_{y=g_1(x)}^{y=g_2(x)}\left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\,dz\right]dy\right]dx$$

→ The order of integration can be changed using the following procedure

Step I Plot the region

→ Plot the curves of variable limit

→ Plot the lines of constant limit

Step II → Identify the constant limits in other variable

→ Identify the lower limit curve, identify the upper limit curve.

Step III → Rewrite the integral.

To identify whether change of variable is needed observe the function and limit curves and see its' eases in other variables.

Change of Variables

$$\boxed{\iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R_{uv}} f(\phi(u,v), \psi(u,v))\,\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv}$$

$x = \phi(u,v)$

$y = \psi(u,v)$

where $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \phi_u & \phi_v \\ \psi_u & \psi_v \end{vmatrix}$

$$\boxed{\begin{aligned} dx\,dy &= r\,dr\,d\phi \\ dx\,dy\,dz &= r\,dr\,d\phi\,dz \\ dx\,dy\,dz &= r^2 \sin\theta\,dr\,d\theta\,d\phi \end{aligned}}$$

* To transform the limits either the region must be plotted or the geometry must be understood

Q 64 (example)

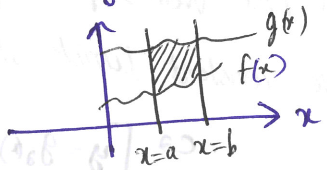
Q 65 example.

Standard problems in geometry

I) Area enclosed by plane curve

⇒ The area of the region R bounded by the curves $y=f(x)$ & $y=g(x)$, $x=a$, $x=b$ in the XY plane is.

$$\boxed{Area = \int_{x=a}^{b}\int_{y=f(x)}^{g(x)} dy\,dx}$$



II) Area In polar coordinates

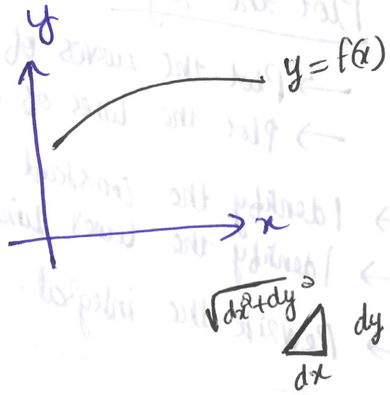
$$\boxed{Area = \iint_R r\,dr\,d\theta}$$

III Length of an arc of a curve

$$length = \int_{x_1}^{x_2} \sqrt{dx^2+dy^2}$$

$$length = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\boxed{Length = \int_{x_1}^{x_2} \sqrt{1+f'(x)^2}\,dx}$$

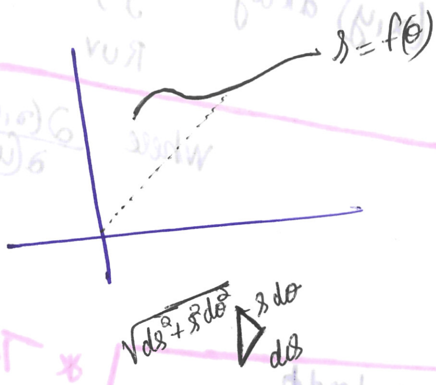


V In polar-coordinates.

$$Length = \int_{\theta_1}^{\theta_2} \sqrt{dr^2+r^2d\theta^2}$$

$$Length = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}\,d\theta$$

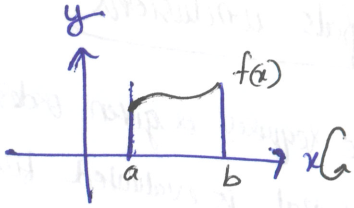
$$\boxed{Lenth = \int_{\theta_1}^{\theta_2} \sqrt{r^2+f'(\theta)^2}\,d\theta}$$ where $r=f(\theta)$



Volumes of Revolution

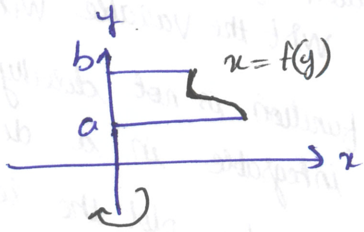
- $$\boxed{V = \int_a^b \pi f^2(x)\, dx}$$

Volume of solid generated by revolving area bounded by curve $y=f(x)$, x axis, lines $x=a$ & $x=b$ about x-axis.



- $$\boxed{V = \int_a^b \pi f^2(y)\, dy}$$

Volume of solid generated by revolving area bounded by curve $x=f(y)$, y axis & lines $y=a$, $y=b$ about y axis.

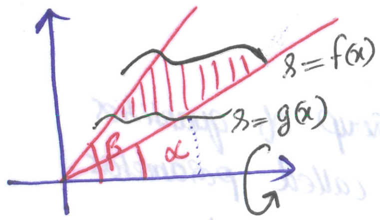


* note if there is lower limit then subtract the volume

* OR subtract the area inside integral

* any way limits are common.

In polar

- 

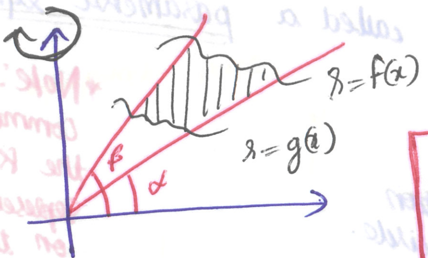
Volume of solid generated by the revolution of the area bounded by the curve $r=f(\theta)$ and $r=g(\theta)$ & lines $\theta=\alpha$, $\theta=\beta$

i) about $\theta = 0°$ line

$$\text{Volume} = \int_\alpha^\beta \int_{g(x)}^{f(x)} \underbrace{2\pi(r\sin\theta)}_{\text{circular length due to rotation}} \underbrace{r\,dr\,d\theta}_{\text{Area element in plane.}}$$

$$\boxed{\text{Volume} = \int_\alpha^\beta \frac{2\pi}{3}\left[f^3(x) - g^3(x)\right]\sin\theta\, d\theta}$$

- ii) about $\theta = 90°$ line



$$\text{Volume} = \int_\alpha^\beta \int_{g(x)}^{f(x)} \underbrace{2\pi r\cos\theta}_{\text{circular length due to rotation}} \underbrace{r\,dr\,d\theta}_{\text{Area element in plane.}}$$

$$\boxed{\text{Volume} = \int_\alpha^\beta \frac{2\pi}{3}\left[f^3(x) - g^3(x)\right]\cos\theta\, d\theta}$$

- Integration of functions with discontinuities or sudden changes must splitted using divide and Rule property.
- eg modulus function, greatest integer function
- 1) First before integration with limits ie, definite integrable see if the function is integrable
  2) If not integral check with various properties one by one.
  3) If still not integrable check application of theorem 1 of integration or Leibnitz rule.

clue – theorem is applicable if the limit has a function of x. ie integration is defined as a function of limit

$$\text{ie } f(x) = \int_{u(x)}^{v(x)} g(t)\,dt.$$

clue – Leibnitz rule is applicable if the integration has an arbitrary constant which can be treated as a second variable.

Multiple integrals conclusions

→ Identify the required or given order of integration by looking at the limit

→ Multiple integral is evaluated first w.r.t the variable whose limits are a function of the remaining variables, that has not been integrated yet and last w.r.t the variable with constant limits.

→ If the given function is not directly integrable in the given order, it might be integrable in a different order in some special cases.

→ In this case try to plot the region and change the order of integration.

→ To identify whether change of variables is needed, observe the funtion and ~~the~~ limit values.

Parametric Equation

→ In mathematics, a parametric equation defines a group of quantities as functions of one or more independent variables called parameters.

→ They are commonly used to express coordinates of points that make up a geometric object (eg curve, surface) in which case the equations are collectively called a parametric representation or parametrization

eg:- $x = \cos t$, $y = \sin t$ } parametric representation of unit circle.

*Note: Parametric equations are commonly used in kinematics where the trajectory of an object is represented by equations depending on time as parameter.

→ Parametrizations are non unique; More than one set of parametric equation can specify the same curve.

Representation of Planar Curves

Type	Form	Example	Description
1) Explicit	$y = f(x)$	$y = mx+b$	Line
2) Implicit	$f(x,y) = 0$	$(x-a)^2+(y-b)^2 = r^2$	Circle
3) Parametric	$x = \frac{g_1(t)}{g_2(t)}$, $y = \frac{h(t)}{h_2(t)}$	$x = a_0 + a_1 t$, $y = b_0 + b_1 t$	Line
		$x = a + r\cos t$, $y = b + r\sin t$	Circle.

Implicitization

→ Converting a set of parametric equation to a single implicit equation involves eliminating the variable t from the simultaneous equations.

→ If any of the equations can be solved for t, then the expression obtained can be substituted an equation involving other quantities only.

Parametric representation of curves in 3D

$$\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

where $\bar{r}(t)$ is the position vector of any point on the curve.

A point is on the curve if it can satisfy this equation for atleast one value of t

ie if (a,b,c) is on curve

then $a = x(t_0)$, $b = y(t_0)$, $c = z(t_0)$ — Atleast one t_0 simultaneously satisfies this.

Differentiability

if $\lim\limits_{\delta t \to 0} \dfrac{\bar{F}(t+\delta t) - \bar{F}(t)}{\delta t}$ exists then differentiable at t.

if $F(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ is a parametric representation of curve C.

then $\boxed{\dfrac{dF}{dt} = \dfrac{d\,x(t)}{dt}\hat{i} + \dfrac{d\,y(t)}{dt}\hat{j} + \dfrac{dz(t)}{d(t)}\hat{k}}$ represents the tangent vector to curve C.

Level surface. [Similar to equipotential points]

if $\phi(x,y,z)$ is a scalar point function/Scalar field then all points (x,y,z) satisfying equation $\boxed{\phi(x,y,z) = c}$ (some constant) is called a level surface of ϕ at level c.

* Note: For different values of c we get different level surfaces and the set of all level surfaces is known as family of level surfaces.

Basic Vector Calculus Results

I) A line containing the point (x_0, y_0, z_0) and parallel to the vector $\bar{v} = a\hat{i} + b\hat{j} + c\hat{k}$ has parametric equation

Equation line.

$$\boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}}$$

if a, b, c are non zero then Symmetric equation of the line can be written as $\Longrightarrow$

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t}$$

Standard form

II) A vector $\vec{N}$ if it is normal to every vector in a plane is called the normal vector to the plane.

Equation of plane.

- Equation of a plane containing the point (x_0, y_0, z_0) with normal vector $\vec{N} = a\hat{i} + b\hat{j} + c\hat{k}$ is

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

$$ax + by + cz = ax_0 + by_0 + cz_0$$

Standard form $\boxed{Ax + By + Cz = D}$ — *Any plane can be expressed as this form

III Angle between 2 planes is the angle between their unit vectors.

Angle btw vectors.

if $\bar{A}$ & $\bar{B}$ are two vectors then the angle between the vectors is given by $\boxed{\theta = \cos^{-1}\left(\frac{\bar{A} \cdot \bar{B}}{|A||B|}\right)}$

IV

Vectors Normal to given vectors

→ if $\bar{A}$ & $\bar{B}$ are two vectors then $\boxed{\frac{\bar{A} \times \bar{B}}{|A||B|}}$ is a unit vector normal to both $\bar{A}$ & $\bar{B}$

⇒ if $\bar{N_1}$ & $\bar{N_2}$ are normal vectors to plane, & plane 2 respectively then the direction vector of intersection of plane must lie in both planes and therefore must be normal to $\bar{N_1}$ & $\bar{N_2}$ simultaneously.

∴ The directional vector giving intersection of two planes with normal $\bar{N_1}$ & $\bar{N_2}$ is $\bar{N_1} \times \bar{N_2}$.

Gradient, curl, Divergence

Definition

For all definition, properties & concepts for these operators refer cmbt book.

☆☆☆

if $Ax+By+Cz=0$ represents a plane

then put $\phi(x,y,z) = Ax+By+Cz = 0$

then $\phi(x,y,z)=0$ represents surface

unit normal to plane $= \boxed{\dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{A\hat{i}+B\hat{j}+C\hat{k}}{\sqrt{A^2+B^2+C^2}}}$

Properties

→ if $\phi(x,y,z)$ is a scalar function then $\nabla\phi$ is in the direction of greatest rate of increase of ϕ and the magnitude is the rate of change (increase of ϕ) in that direction

→ $\nabla\phi$ at a point $P(x_0,y_0,z_0)$ is normal to the surface given by $\phi(x,y,z)=c$

→ $\phi(x,y,z)=c$ represents the surface or set of all points where ϕ is same

→ This is because $\phi(x,y,z)=c$ represents the surface or set of all points where ϕ is same

∴ There will be no rate of change of ϕ in the tangential direction to this surface.

∴ Since $\nabla\phi$ is along greatest rate of increase, no component of $\nabla\phi$ is along tangent to $\phi(x,y,z)=c$ & the greatest rate of change is achieved normal to this surface.

→ Directional derivative of ϕ in the direction of given vector $\bar{a}$ $= \boxed{\nabla\phi \cdot \dfrac{\bar{a}}{|\bar{a}|} \text{ or } \nabla\phi\cdot\hat{a}}$

Very important unit vector

→ Max rate of change of $\phi(x,y,z)$ at $P(x_0,y_0,z_0)$ $= \boxed{|\nabla\phi|_{@P}}$

→ If $\phi_1(x,y,z)=c_1$ & $\phi_2(x,y,z)=c_2$ are two surfaces then at their point of intersection $P(x_0,y_0,z_0)$

Let θ be the angle b/w two surfaces

$$\boxed{\theta = \cos^{-1}\left[\frac{(\nabla\phi_1)\cdot(\nabla\phi_2)}{|\nabla\phi_1|\,|\nabla\phi_2|}\right]}$$

- Equation of tangent plane to the surface $\phi(x,y,z)=c$ @ a point $P(x_0,y_0,z_0)$ with $(\nabla\phi)_{@P} = a\hat{i}+b\hat{j}+c\hat{k}$

then $\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$

* Since $\nabla\phi$ at P is normal to tangent plane and P is a point in plane directly result can be used.

‼ Equation of normal line to the surface $\phi(x,y,z)=c$ @ point $(P) = P(x_0,y_0,z_0)$ with $\nabla\phi_{@P} = a\hat{i}+b\hat{j}+c\hat{k}$

is $= \boxed{\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}}$

* Since $\nabla\phi$ at P is a vector parallel to required line & P is a point on line the direct result can be applied.

Miscallaneous ★★★★★

- $\boxed{\text{If } \bar{F}(t) \text{ is a vector with constant Magnitude then } \bar{F}\cdot\frac{d\bar{F}}{dt}=0}$.

proof

Let $\bar{F}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$

since $|\bar{F}(t)| = \text{constant}$ $\quad x^2(t) + y^2(t) + z^2(t) = c$

$$2x(t)\frac{dx(t)}{dt} + 2y(t)\cdot\frac{dy(t)}{dt} + 2z(t)\cdot\frac{dz(t)}{dt} = 0$$

$$\Rightarrow \boxed{x(t)\cdot\dot{x}(t) + y(t)\,\dot{y}(t) + z(t)\cdot\dot{z}(t) = 0}$$

now. $\dot{\bar{F}}(t) = \frac{d}{dt}\bar{F}(t) = \dot{x}(t)\,\hat{i} + \dot{y}(t)\,\hat{j} + \dot{z}(t)\,\hat{k}$

$$\therefore \bar{F}\cdot\frac{d\bar{F}}{dt} = x\dot{x} + y\dot{y} + z\dot{z} = 0$$

- If $\bar{F}(t)$ is a vector with constant direction then $\bar{F}\times\frac{d\bar{F}}{dt} = 0$

* if $\bar{F}(t)$ has constant direction $\frac{d}{dt}F(t)$ has no component ~~part~~ perpendicular to $F(t)$

$\Rightarrow \bar{F}(t)$ & $\frac{d}{dt}F(t)$ is codirectional $\Rightarrow \bar{F}(t)\times\frac{d}{dt}F(t) = 0$

* Similarly if $\bar{F}(t)$ has constant magnitude $\frac{d}{dt}F(t)$ has no component parallel to $F(t)$

$\Rightarrow F(t)$ & $\frac{d}{dt}F(t)$ are perpendicular $\Rightarrow F(t)\cdot\frac{d}{dt}F(t) = 0$

Misc points

- The divergence of the velocity field of a incompressible fluid is zero.
- The velocity potential of a ideal fluid ϕ is harmonic $\Rightarrow \nabla^2\phi = 0$

Proof: Let Vector potential $= \phi$

then $\nabla\phi = $ velocity field

but $\nabla\cdot(\nabla\phi) = 0$ [incompressible]

$\Rightarrow \nabla^2\phi = 0$

- Harmonic field — Laplacian is zero.

Vector integration

$$d\bar{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$d\bar{s} = \hat{n}\,ds = dydz\,\hat{i} + dzdx\,\hat{j} + dxdy\,\hat{k}$$

$$dv = dxdydz$$

Complex integration $\boxed{dz = dx + i\,dy \quad i = \sqrt{-1}}$

Line integral can be splitted to separate integrals if path is simple.

Line integral

- Let $\bar{A}(x,y,z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ be a differentiable vector function.
- Then the integral of tangential component of $\bar{A}$ along c from P_1 to P_2 given by

$$\boxed{\oint_{P_1}^{P_2} \bar{A}\cdot dl = \int_{P_1}^{P_2} A_1 dx + A_2 dy + A_3 dz}$$

$$\boxed{\bar{dl} = dx\hat{i} + dy\hat{j} + dz\hat{k}}$$

Circulation — closed path — counterclockwise — +ve.

- If $\bar{A}$ is a force vector acting on a particle which moves from P_1 to P_2 along curve C then $\oint_{P_1}^{P_2} \bar{A}\cdot dl$ gives total work done by Force $\bar{A}$.
- To get work done by particle against force A to move from P_1 to P_2 $= -\int_{P_1}^{P_2} \bar{A}\cdot dl$

If $\bar{A}$ is conservative or irrotational then $(\nabla\times A = 0)$

$\Rightarrow$ line integral $\int_{P_1}^{P_2} \bar{A}\cdot dl$ is independent of path C.

$$\int_{P_1}^{P_2} \bar{A}\cdot ds = \int_{P_1}^{P_2} \nabla\phi\cdot ds = \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1)$$

where $A = \nabla\phi$

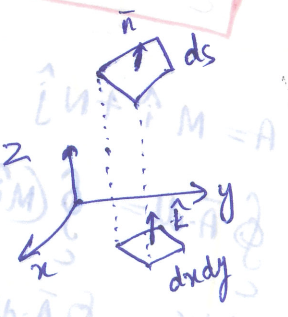
* since every irrotational field can be expressed as gradient of scalar field. $(\nabla\times\nabla\phi) = 0$ gradient is irrotational.

$$\boxed{\text{Als. if } \nabla\times A = 0 \quad \oint \bar{A}\cdot dl = 0}$$

Surface integral.

- $\iint_S (\bar{F}\cdot\bar{n})ds = \iint_{R_1} (\bar{F}\cdot\bar{n}) \dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$
- $\iint_S (\bar{F}\cdot\bar{n})ds = \iint_{R_2} (\bar{F}\cdot\bar{n}) \dfrac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$
- $\iint_S (\bar{F}\cdot\bar{n})ds = \iint_{R_3} (\bar{F}\cdot\bar{n}) \dfrac{dx\,dz}{|\bar{n}\cdot\bar{j}|}$

R_1 projection of S on xy plane
R_2 projection of S on yz plane
R_3 projection of S on xz plane.



$ds = \dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

$ds = \dfrac{dx\,dy}{\cos\theta}$

θ ∠ between $\bar{n}$ & $\bar{k}$

* $\hat{n}$ is the unit vector normal to surface ds

$|\hat{n}\cdot\hat{k}| = |\hat{n}||\hat{k}|\cos\theta = \cos\theta$

Method to evaluate vector volume & surface integrals

surface integral

$$\iint_S (\bar{F}\cdot\bar{n})\,ds = \iint_{R_1} \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\hat{k}|}$$

Step I → Find unit vector normal to surface $\bar{n}$

Step II → Find & evaluate $(\bar{F}\cdot\hat{n})$

Step III Find projection of surface S on ~~only~~ any coordinate plane

Step IV Find limits of plane variables from the plane equation.

Step V Find expression of Non plane variable.

Step VI Now surface integration is converted to double integration.

See example 85 & 86.

Taking limit from plane equation

eg) $2x+3y+6z=12$ → projection of the plane surface that exist in first quadrant to xy plane.

$x>0,\ y>0\ z>0$ $\quad z = 0 \text{ to } \left(\frac{12-2x-3y}{6}\right)$

x, y limit got by putting $z=0$

$2x+3y=12$ $\quad y = 0 \text{ to } \frac{12-2x}{3}$

x limit got by putting $y=0$

$2x=12$

$x=6,0$

eg) $\int_V dv$ V is closed

region bounded by $x=0, y=0, z=0$

$2x+2y+z=4$ — plane.

$z = 4-2x-2y$

$y = \frac{4-2x}{2}$ (at $z=0$)

$x = \frac{4}{2} = 2$

Green's theorm

Q — If R is a closed Region of the xy plane bounded by C [Closed curve] if $M[x,y]$ & $N[x,y]$ $\frac{\partial N}{\partial x}, \frac{\partial M}{\partial y}$ are continuous functions of x, y in R. then

$$\boxed{\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

Proof

Consider a vector $A = M\hat{i} + N\hat{j}$

then $\oint_C \bar{A}\cdot d\bar{l} = \oint_C (M\hat{i}+N\hat{j})\cdot(dx\,\hat{i}+dy\,\hat{j}+dz\,\hat{k}) = \oint_C M\,dx + N\,dy$

But According to stokes $\oint_C \bar{A}\cdot d\bar{l} = \iint_S \nabla\times A\cdot ds$ $\quad \nabla\times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix}$

$\nabla\times A = \hat{k}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$

$$\oint_C \bar{A}\cdot dl = \int_S \nabla\times\bar{A}\cdot ds \quad \int_S \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\hat{k}\left[dydz\,\hat{i}+dzdx\,\hat{j}+dxdy\,\hat{k}\right]$$

$$\oint_C A\cdot dl = \boxed{\oint_C M\,dx+N\,dy = \iint_S\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy}$$

Gauss Divergence Theorm

$$\oiint_S \bar{A}\cdot\bar{n}\,ds = \iiint_V (\nabla\cdot\bar{A})\,dV$$

Stokes theorm

$$\oint_C \bar{A}\cdot dl = \iint_S (\nabla\times\bar{A})\,\bar{ds}$$

$\bar{r}$ is the position vector of a point

$\Rightarrow \bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$

eg) $\oiint_S \bar{r}\cdot ds =$

a) $\frac{1}{2}V$

b) V

c) $2V$

d) $3V$

$S\to$ closed surface

$V\to$ closed volume bys

$\bar{r}\to$ position vector

Solution $\oiint_S \bar{r}\cdot ds = \iiint_V \nabla\cdot\bar{r}\,dV$ [Gauss divergence theorm]

$r = x\hat{i}+y\hat{j}+z\hat{k}$

$\nabla\cdot r = 1+1+1=3$

$\Rightarrow 3\iiint_V dV = \underline{3V}$ option d

Fouries Series

Standard form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega x) + b_n\sin(n\omega x)$$

period $= T$

$\omega = \frac{2\pi}{T}$

$$a_0 = \frac{1}{T}\int_0^T f(x)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cos(n\omega t)\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(x)\sin(n\omega t)\,dt$$

Even $\to$ only cos + DC

Odd $\to$ only sine + No. DC

Half wave $\to$ No odd harmonies

$a_0 =$ Average over period

$$\boxed{a_0 = \frac{\text{Area}}{T}}$$

Dirichlet's condition

- $f(x)$ is periodic, single valued & finite
- $f(x)$ has a finite number of discontinuities in any one period.
- $f(x)$ has at the most a finite no. of max & minima

Sufficient conditions.

Half Range Series

- Only half of the function will be given out of total time period
- ~~Rem~~ Rest half is taken based on whether half range sine or half rang cosine.

Half Range Sine

- Given function $f(x)$ defined in $(0, l)$

put $\boxed{T = 2l}$

$\boxed{\begin{array}{l} \cancel{f(x)} = \cancel{f(x)} \quad \text{Sin} \Rightarrow \text{Odd} \\ f(-x) = -f(x) \end{array}}$

$\omega = \frac{2\pi}{2l}$

$\omega = \frac{\pi}{l}$

$\therefore$ $a_0 = 0$

$a_n = 0$

$b_n = 2 \times \frac{2}{T} \int_{-l}^{l} f(x) \sin(n\omega t) dt$

$b_n = \frac{2 \times 2}{T} \int_{0}^{l} f(x) \sin(n\omega t) dt$

$$\boxed{b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin(n\omega t) dt}$$

Half Range Cosine

- Given function $f(x)$ defined in $(0, l)$

$\boxed{\text{put } T = 2l}$

$\boxed{f(x) = f(-x)}$ Cos $\Rightarrow$ even

$a_0 = \frac{1}{l} \int_{0}^{l} f(x) dx$ $\left[\frac{\text{Area}}{T}\right]$

$a_n = \frac{2}{T} \int_{-l}^{l} f(x) \cos(n\omega t) dt$

$$\boxed{a_n = \frac{2}{l} \int_{0}^{l} f(x) \cos(n\omega t) dt.}$$

- Average of sin or cosine over a period is zero
- ~~[illegible]~~ Different frequency sinusoidals are orthogonal
- $\cos n\pi = \sin\left(\frac{2n+1}{2}\right)\pi = (-1)^n$
- If $f(x)$ has a discontinuity (at $x = a$) then fourier series converges to $\boxed{\frac{f(a^-) + f(a^+)}{2}}$ at $x = a$ ie Average of left hand and right limit.

★ This can be used to find the convergence of infinite series. example 91, 92, 99.

Note Worthy Misc

○ If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ ie position vector and $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

then $\nabla\phi(r) = \hat{r}\,\phi'(r)$

proof $\nabla\phi(r) = \nabla\phi\left(\sqrt{x^2+y^2+z^2}\right)$

$= \phi'\left(\sqrt{x^2+y^2+z^2}\right) \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}}\left(2x\hat{i} + 2y\hat{j} + 2z\hat{k}\right)$

$\boxed{\text{See ex 65}}$

$= \phi'(r)\,\frac{\bar{r}}{r} = \underline{\phi'(r)\cdot\hat{r}}$

○ $\nabla\cdot(\phi\bar{A}) = \boxed{\phi(\nabla\cdot\bar{A}) + \bar{A}\cdot(\nabla\phi)}$ — Identity.

eg) if $\bar{r}$ is position vector then find div $(e^r\bar{r})$

$\nabla\cdot(e^r\bar{r}) = \bar{r}\cdot(\nabla(e^r)) + e^r(\nabla\cdot\bar{r})$

$= \bar{r}\cdot(e^r(\hat{r})) + e^r\cdot 3$

$= e^r\,\frac{\bar{r}\cdot\bar{r}}{r} + e^r\cdot 3$

$= e^r\,\frac{r^2}{r} + e^r\cdot 3 = \underline{e^r(r+3)}$

○ $\boxed{\nabla\times(\phi\bar{A}) = \cancel{\bar{A}\times\nabla\phi}\ \phi(\nabla\times A) + (\nabla\phi\times A)}$ Identity

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

eg) If r is position vector then curl $(r^4\bar{r})$

$\nabla\times(r^4\bar{r}) = r^4(\nabla\times\bar{r}) + \nabla(r^4)\times\bar{r}$

$= r^4(0) + 4r^3\frac{\bar{r}}{r}\times\bar{r}$

$r^4(0) + 4r^2\,\bar{r}\times\bar{r} = \underline{0}.$

☆ Equation of line in 3D from point x_0, y_0, z_0 to x_1, y_1, z_1

is $\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0} = t$

2D $\frac{x-x_0}{y-y_0} = \frac{x_1-x_0}{y_1-y_0}$

$\int_C \bar{f}\cdot d\bar{s}$ alonge straight line joining P_1, P_2 $\quad P_1 = x_1 y_1 z_1$, $P_2 = x_2 y_2 z_2$

$f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$

CASE I

f Not conservative

then → obtain equation of line

$$\int_C f_x\,dx + \int f_y\,dy + \int f_z\,dz$$

- either change dy to dx, dz to dx or vice verse
- or substitute for z, y in terms of x or vice verse
- limit for the integrating variable only

CASE II

f conservative

then

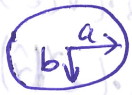
$$\int_{P_1}^{P_2} f\,dl = \int_{x_1y_1z_1}^{x_2y_1z_1} f_x\,dx + \int_{x_2y_1z_1}^{x_2y_2z_1} f_y\,dy + \int_{x_2y_2z_1}^{x_2y_2z_2} f_z\,dz$$

$\boxed{y \text{ const},\ z \text{ const}}$ $\quad \boxed{x, z \text{ constant}}$ $\quad \boxed{x, y \text{ const}}$

- or find ϕ such that $\nabla\phi = f$

then $\int_{P_1}^{P_2} \bar{f}\cdot d\bar{l} = \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi P_1$

- Area of ellipse $= \pi ab$ $\quad a \to \frac{\text{principle dia}}{2}$ $\quad b \to \frac{\text{minor dia}}{2}$



$f_1(x,y,z)\,dydz + f_2(x,y,z)\,dzdx + f_3(x,y,z)\,dxdy$

$$\iint_S f_1(x,y,z)\,dydz + \ldots$$

where $d\bar{s} = \cancel{dx\hat{i} + dy\hat{j} + dz\hat{k}}$ $dydz\,\hat{i} + dzdx\,\hat{j} + dxdy\,\hat{k}$

$$= \iint_S (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})\cdot d\bar{s}$$

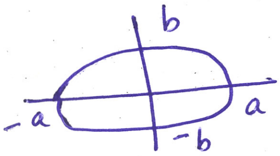
$$= \iiint_V (\nabla \times F)\,dV \quad \text{where } F = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$$

Area of ellipse proof

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$y=0 \Rightarrow x = \pm a$

$x=0 \Rightarrow y = \pm b$



$y = \pm b\sqrt{1-\frac{x^2}{a^2}}$

$$\text{Area} = \int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy\,dx = \int_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}\,dx = \int_{-a}^{a}\frac{2b}{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{2b}{a}\left[\cancel{\frac{x}{2}\sqrt{a^2-x^2}}^{\,0} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{-a}^{a}$$

$$= \frac{2b}{a}\left[\frac{a^2}{2}\left[\pi/2 - (-\pi/2)\right]\right] = \frac{\cancel{2}b}{\cancel{a}}\cdot\frac{a^{\cancel{2}}}{\cancel{2}}\cdot\pi = \underline{\pi ab}$$